J. Fluid Mech. (1967), *vol.* 27, *part* 4, *pp.* 657–689
Printed in Great Britain

On the stability of an inviscid shear layer which is periodic in space and time

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(Received 8 November 1965 and in revised form 24 May 1966)

In experiments concerning the instability of free shear layers, oscillations have been observed in the downstream flow which have a frequency exactly half that of the dominant oscillation closer to the origin of the layer. The present analysis indicates that the phenomenon is due to a secondary instability associated with the nearly periodic flow which arises from the finite-amplitude growth of the fundamental disturbance.

At first, however, the stability of inviscid shear flows, consisting of a non-zero mean component, together with a component periodic in the direction of flow and with time, is investigated fairly generally. It is found that the periodic component can serve as a means by which waves with twice the wavelength of the periodic component can be reinforced. The dependence of the growth rate of the subharmonic wave upon the amplitude of the periodic component is found for the case when the mean flow profile is of the hyperbolic-tangent type. In order that the subharmonic growth rate may exceed that of the most unstable disturbance associated with the mean flow, the amplitude of the streamwise component of the periodic flow is required to be about 12 % of the mean velocity difference across the shear layer. This represents order-of-magnitude agreement with experiment.

Other possibilities of interaction between disturbances and the periodic flow are discussed, and the concluding section contains a discussion of the interactions on the basis of the energy equation.

1. Introduction

In his experiments concerning the transition to turbulence of a separated shear layer, Sato (1956, 1959) confirmed that results obtained through consideration of the stability of unbounded shear flows on the basis of linear inviscid stability theory could be used to describe the actual onset of oscillations. The agreement between theory and experiment was restricted, however, to a region relatively near the origin of the layer (which was the edge of a step along whose upper surface existed a laminar flow). Further downstream, not only a harmonic component, with twice the frequency of the fundamental, but also a subharmonic component, with half the frequency, appeared. Evidence of this phenomenon is

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given in figure 1, which was taken from Sato's (1959) paper. The simplest non-linear argument would lead one to expect the existence of the harmonic wave; the origin of the subharmonic wave, of order one-half, is less evident.

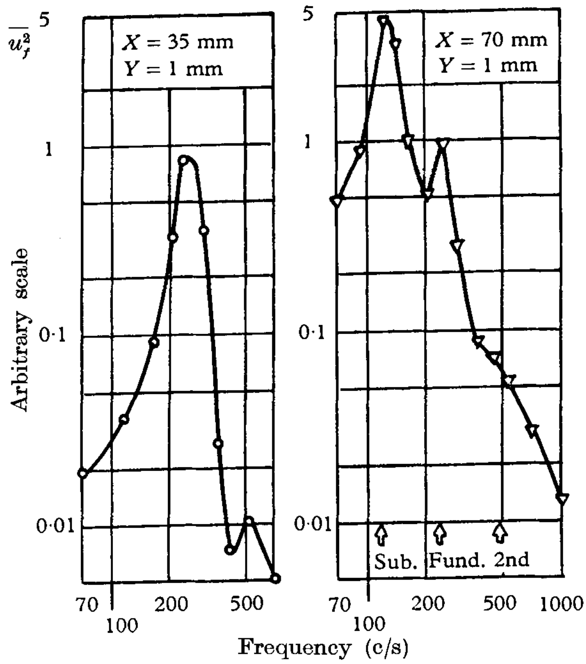

FIGURE 1. Energy spectrum of fluctuation in the streamwise component of velocity, measured at two stations downstream of the step (figure 8 of Sato 1959).

The thickness of the shear layer increases, of course, with distance downstream from the step. However, any attempt to explain the origin of the subharmonic on the basis of the growth of shear-layer thickness would encounter difficulty in explaining why Sato & Kuriki (1961) found no such subharmonic waves in the transition of the wake behind a flat plate. Although similar downstream growth of the shear-layer thickness occurred, only the fundamental and second harmonic components were observed. The purpose of the present analysis is to show that a mechanism exists for the generation of a subharmonic wave, at least in the case of a flow with a hyperbolic tangent velocity profile. This flow is quite similar to that investigated by Sato. The mechanism arises directly from the finite-amplitude growth of the fundamental instability.

Wille (1963) also observed the development of subharmonic waves while investigating the stability of both circular and plane jets (cf. figures 25 and 53 of his report and also figure 5 of the article by Wehrmann & Wille (1958)), as did Bradshaw (1966) in the case of a circular jet. On the basis of flow visualization with smoke (figure 1 of his report), Wille remarked that the occurrence of the subharmonic could be identified as a fusion of the vortices which tend to form during the late stages of transition. The formation of these vortices from the initial instabilities has been discussed recently by Michalke (1965a). In this paper, we shall base our arguments upon the vorticity associated with the fundamental disturbance and shall not use any discrete vortex model; the coalescence concept is, however, undoubtedly a graphic way of describing the problem and will be discussed further in the concluding section of this paper.

Because the non-linear self-interaction of a wave will produce only waves of shorter wavelength, a subharmonic wave can arise only through the interaction of waves of different wavelengths. One could proceed with a formal non-linear analysis on this basis, in much the same manner as Segel (1962) with regard to the thermal convection problem. A few comments apropos of this approach will be made later. However, careful reading of the experimental data coupled with some knowledge concerning the non-linear self-interaction of the fundamental for the flow of interest (cf. Schade 1964) allow us to construct a reasonable model of the flow just before the emergence of the subharmonic. It is hoped that this model permits not only simpler analysis but also a deeper insight into the physical processes of importance.

In figure 1, the energy spectrum of the fluctuation in velocity is shown for two downstream stations. Before the subharmonic has emerged, the energy is strongly peaked at the fundamental frequency with a much smaller peak at the frequency of the second harmonic. After the subharmonic has emerged, the peak value of energy occurs at the subharmonic frequency. A relative peak still exists, however, at the frequency of the fundamental, and the magnitude of the energy has nearly the same value as at the earlier station. This indicates that while the flow, taken as a whole, is still evolving the fundamental instability has achieved a state close to equilibrium. Schade (1964) has investigated the non-linear self-interaction of the neutral disturbance associated with the flow which has a hyperbolic-tangent velocity profile. By extrapolation of the calculated value of the second Landau constant, he estimated that the equilibrium amplitude of the most strongly amplified disturbance would be about 17 % of the velocity difference across the shear layer. Sato (1959) listed the root-mean-squared values of the velocity fluctuations at the two downstream stations noted in figure 1 as approximately 3 and 10 % of the velocity difference across the shear layer (see figure 17 of his paper). Thus, the actual magnitude of the fundamental just before the emergence of the subharmonic lies between these last two values. These figures indicate that the fundamental may grow to an appreciable size relative to the mean flow.

Our model is based upon the above comments and is as follows. Linear, inviscid stability theory is assumed to give the correct wave-number of the initially most unstable disturbance. We then appeal to a non-linear mechanism which selects this wave-number for amplification and causes the disturbance energy to be more sharply centred at this wave-number than might be expected on the basis of linear theory (cf. Segel (1962) for a discussion of this mechanism in the thermal convection problem). The disturbance grows rapidly relative to waves whose wave-numbers are near to its own until it approaches a state close to finite-amplitude equilibrium. We then have a picture of a flow composed of the basic, parallel component and a non-parallel component, whose nearly periodic behaviour with time and in the direction of flow are characterized by, respectively, the frequency and wave-number of the fundamental instability.

The above flow occurs only for a certain value of the amplitude of the primary disturbance. However, we shall regard this amplitude as a parameter and consider how the magnitude of the periodic component affects the growth rate of disturbances associated with the mean, parallel flow component. It will be shown later that the periodicity exerts its greatest influence upon a wave with half of its own wave-number. While this result can be established without specifying the details of the periodic flow, we shall apply the result to experiment by setting the wave-number of the periodic flow equal to that of the fundamental. We shall then calculate the amplitude of the periodicity required to make the growth rate of the subharmonic exceed that of the most unstable disturbance associated with the mean, parallel flow. If this critical amplitude compares reasonably well with the measured values of the fundamental before the emergence of the subharmonic, we shall conclude that the periodic flow associated with the fundamental can serve as a means by which further energy can be extracted from the mean flow and fed into a wave of longer wavelength.

2. Formulation of the stability problem

We define a stream function $\psi(x, y, t)$ so that

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x, \tag{2.1}$$

where x is in the direction of the mean flow, y is normal to that direction, and u and v are the corresponding velocity components. We shall assume that the flow is two-dimensional. Although Sato (1959) states that the fluctuations in the non-linear region were of a three-dimensional type, he does not present any data which could be employed in our model to reflect this feature. The more recent results of Browand (1966), however, show that the flow retains its two-dimensionality during the emergence of the subharmonic oscillation. The results of this paper indicate that the emergence of the subharmonic can be predicted quite well on the basis of a two-dimensional theory.

For the moment, we consider a basic flow $\psi_0(x, y, t)$ which is assumed to be a solution of the equations of motion. Thus $\psi = \psi_0(x, y, t)$ satisfies

$$\left\{\frac{\partial}{\partial t}+\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right\}\nabla^2\psi = 0. \tag{2.2}$$

We now assume that this flow is perturbed by a disturbance of $O(\epsilon)$ and attempt to find a solution by expanding as

$$\psi(x, y, t) = \psi_0(x, y, t)+\epsilon\psi_1(x, y, t)+O(\epsilon^2). \tag{2.3}$$

The perturbation stream function of $O(\epsilon)$ then satisfies

$$\left[\frac{\partial}{\partial t}+\frac{\partial\psi_0}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi_0}{\partial x}\frac{\partial}{\partial y}\right]\nabla^2\psi_1+\left(\frac{\partial}{\partial x}\nabla^2\psi_0\right)\frac{\partial\psi_1}{\partial y}-\left(\frac{\partial}{\partial y}\nabla^2\psi_0\right)\frac{\partial\psi_1}{\partial x} = 0. \tag{2.4}$$

We note two features of (2.4) which are peculiar to the case $\partial\psi_0/\partial x \neq 0$. First, the equation is of the third order with respect to y and would therefore seem to require three boundary conditions, in contrast to the two required for the case of a parallel basic flow. Secondly, the term $\partial\psi_0/\partial x$ multiplies the highest derivative with respect to y. Hence, if an expansion is attempted which regards the amplitude of the basic periodic flow as small in some sense, the expansion may well be of a singular-perturbation type.

However, in this paper, we shall consider only unbounded shear flows. The linear stability characteristics of such flows are predicted quite well by inviscid flow analyses. The boundary conditions which are imposed when $\psi_0 = \psi_0(y,t)$ are

$$\partial\psi_1/\partial x \to 0 \quad \text{as} \quad y \to \pm\infty, \tag{2.5}$$

which imply that $\psi_1 \to 0$ as $y \to \pm\infty$ for disturbances periodic in the flow direction. While these two conditions are sufficient to obtain a solution, at least when the basic velocity profile has a point of inflexion, we should realize that a plausible consequence in the problem considered of the above boundary conditions is that

$$\partial\psi_1/\partial y \to 0 \quad \text{as} \quad y \to \pm\infty. \tag{2.6}$$

But these are the additional boundary conditions which would be applied in the viscous problem. In other words, the inviscid solution for the case of a free shear layer satisfies the boundary conditions applied to the viscous solution. We shall approach (2.4) from this point of view and presume that the solutions obtained through application of (2.5) will also satisfy (2.6). Thus, the question of a third boundary condition suitable to (2.4) is not considered further.

Viscosity will also be invoked to remedy any singular behaviour resulting from a perturbation analysis. The governing equation would not be a singular-perturbation type with regard to $\partial\psi_0/\partial x$ if the viscous terms were included. Schade (1964) also found that viscosity had to be included at the critical layer in his otherwise inviscid, non-linear analysis.

We now consider a case for which $\psi_0(x,y,t)$ may be expressed by an expansion such as

$$\psi_0(x,y,t) = \psi_{00}(y) + \delta\psi_{01}(x,y,t) + O(\delta^2), \tag{2.7}$$

in which $|\delta| \ll 1$ but $|\delta| \gg |\epsilon|$. Thus, we shall assume that the amplitude of the disturbance is small compared to the magnitude of the periodicity. Now, (2.7) may represent the expansion in terms of δ of any exact solution to the inviscid flow equations. For instance, Stuart (1966) has found such an exact solution in the form

$$\psi_0(x,y) = \ln[C\cosh y + (C^2-1)^{\frac{1}{2}}\cos x], \tag{2.8}$$

where C is an arbitrary constant. Thus, (2.7) might represent the expansion of this solution for moderate values of $(C^2-1)^{\frac{1}{2}}/C$ (in the limit $C^2 \to 1+\delta^2$, $\delta^2 \ll 1$, the $O(\delta)$ term in the expansion of (2.8) is the neutral eigensolution associated with the flow $\psi_{00} = \ln\cosh y$).

In this paper, (2.7) will be later taken to represent the flow resulting from finite-amplitude growth of the primary disturbance associated with the parallel flow $U_{00}(y) = \tanh y$. Schade (1964) has considered the nature of such growth by expanding

$$\begin{aligned} u_0(x,y,t) = u_{00}(y) + |A|^2 u_{00}^{(1)}(y) + \ldots + A(u_{01}(y) + |A|^2 u_{01}^{(1)}(y) + \ldots)e^{i\alpha x} \\ + (A^2 u_{02}(y) + \ldots)e^{2i\alpha x} + \ldots + \text{complex conjugates}, \end{aligned} \tag{2.9}$$

where $A = A(t)$. On the basis of the previously mentioned experimental evidence, we shall assume that $A(t)$ tends to a constant as $t \to \infty$ when α has the value of the initially most unstable wave. Now $u_{01}(y)$ denotes the eigensolution corresponding to the linear stability problem. Because representative values of $|A|$ are 0·1–0·2, we shall neglect terms of $O(\delta^2) \sim O(|A|^2)$ in our analysis and assume that the flow can be modelled by superimposing $u_{01}(y)\exp(i\alpha x)$ together with its conjugate upon the basic parallel flow. The analysis could be extended, however, to include the higher-order terms, and a conjecture concerning their effect will be made in the appropriate place.

While the amplitude $|A|$ in (2.9) actually has a numerical value as $t \to \infty$, we shall regard δ in (2.7) as a parameter in order to investigate the stability of (2.7) by means of a perturbation procedure. Thus, (2.7) will not be a solution of the equations of motion for all values of δ. However, we shall interpret the results for a certain value of δ when (2.7) represents the flow (2.9) and is therefore assumed to be asymptotic to a solution of the equations of motion.

An approach similar to this has been used by Greenspan & Benney (1963) in their work concerning the onset of turbulence in boundary-layer flow along a flat plate. They considered the stability of an unsteady, parallel flow shear profile. By itself, this flow would not be a solution of the equations of motion. However, it was taken to be representative of the flow observed experimentally just before the onset of turbulence and therefore also of *some* solution to the equations of motion. The results were interpreted by assigning empirical values to the various parameters entering into their analysis. Similarly, we shall first consider the stability of (2.7) in a rather general manner, and then consider a special case for which this flow has some meaning on the basis of both theory and experiment.

When $\psi_0(x,y,t)$ is given in the form (2.7), we can consider obtaining a solution to (2.4) by expanding

$$\psi_1(x,y,t) = \psi_{10}(x,y,t) + \delta\psi_{11}(x,y,t) + O(\delta^2), \tag{2.10}$$

where each term must satisfy (2.5). Thus, $\psi_{10}(x,y,t)$ represents any solution of the linear stability equation

$$\left(\frac{\partial}{\partial t} + \psi'_{00}\frac{\partial}{\partial x}\right)\nabla^2\psi_{10} - \psi'''_{00}\frac{\partial\psi_{10}}{\partial x} = 0, \tag{2.11}$$

in which a prime denotes differentiation with respect to y. We assume that $\psi'''_{00}(y)$ vanishes for some value of y so that there are solutions to (2.11) of the form

$$\psi_{10}(x,y,t) = B\phi_{10}(y)\exp\{i\alpha_0(x-c_0t)\} + \tilde{B}\tilde{\phi}_{10}(y)\exp\{-i\alpha_0(x-\tilde{c}_0t)\}, \tag{2.12}$$

where a tilde denotes the complex conjugate and B is an arbitrary constant.

It should be remarked that in assuming (2.12) as the zero-order solution of (2.4) we also assume that the terms of $O(\delta)$ and higher in (2.7) merely shift and do not eliminate the inflexion point associated with $\psi_{00}(y)$. The importance of this restriction can be realized by considering the special case $\psi_{01} = \psi_{01}(y)$. We then know that there are no eigensolutions which grow with time if $\psi'''_0(y)$ does not vanish for some value of y. The perturbation procedure used in this analysis would be invalid for the rather special case when the higher-order terms in (2.7) eliminate the inflexion point of the velocity profile.

The general characteristics of the solutions (2.12) for various parallel flows are known (cf. Drazin & Howard (1962) for a survey). However, not many solutions have been calculated in detail, and it is clear that we must have detailed knowledge concerning the eigensolutions in order to proceed further. One case for which quite specific knowledge is available and which is similar to the separated shear flow is the flow given by

$$U_{00} = \tanh y. \tag{2.13}$$

The stability characteristics of this flow with respect to disturbances which grow with time have been calculated by Betchov & Szewczyk (1963) and by Michalke (1964). In the former analysis, the authors found that the phase velocity, $c_{0,r}$, was zero for their eigensolutions for a wide range of Reynolds numbers. This fact tends to confirm the result of Tatsumi & Gotoh (1960) that the phase velocity is zero for all disturbances which grow with time in the case of antisymmetric shear flows. Tatsumi & Gotoh essentially showed that, if a solution exists for some non-zero value of $c_{0,r}$, a second solution must exist for the same value of wave-number and with the same growth rate but with $c_{0,r}$ of the opposite sign to that of the first solution. Such solutions have not been found for growing disturbances in the case of antisymmetric shear flows with one inflexion point, although Gallagher & Mercer (1962) have found such solutions which decay with time in the case of plane Couette flow. Hence, the conclusion that $c_{0,r} \equiv 0$ for such flows in the case of temporally growing disturbances seems justified. On the other hand, a similar conclusion would seem unwarranted for the case of spatially growing disturbances. It would then seem reasonable to expect that, if a solution exists with $c_{0,r} > 0$ and which grows, or decays, in the positive x-direction, a second solution should exist with $c_{0,r} < 0$ which grows, or decays, respectively, in the negative x-direction. The question of whether such solutions exist for the present flow is discussed in the appendix to this paper. It is concluded there that such solutions exist but that they are damped in the respective directions. Because of this result, we shall proceed on the basis that the eigensolutions corresponding to temporally growing disturbances describe completely the unstable behaviour of the flow (2.13). At a later stage, we shall consider the velocity profile given by

$$U_0 = 0{\cdot}5(1+\tanh y), \tag{2.14}$$

which is closer to the separated shear-layer profile and for which the disturbances have a non-zero phase velocity. We shall then discuss whether the results of our temporal growth analysis can be related to the behaviour of spatially growing disturbances which are observed in experiments concerning separated shear layers.

The equation for $\psi_{11}(x,y,t)$ is

$$\left[\frac{\partial}{\partial t}+\psi'_{00}\frac{\partial}{\partial x}\right]\nabla^2\psi_{11}-\psi'''_{00}\frac{\partial\psi_{11}}{\partial x} = -\left[\frac{\partial\psi_{01}}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi_{01}}{\partial x}\frac{\partial}{\partial y}\right]\nabla^2\psi_{10} -\left(\frac{\partial}{\partial x}\nabla^2\psi_{01}\right)\frac{\partial\psi_{10}}{\partial y}+\left(\frac{\partial}{\partial y}\nabla^2\psi_{01}\right)\frac{\partial\psi_{10}}{\partial x}. \tag{2.15}$$

It is clear that the form of the solution for $\psi_{11}(x,y,t)$ will depend greatly upon how $\psi_{01}(x,y,t)$ and $\psi_{10}(x,y,t)$ interact, and we shall now discuss the possible interactions in detail.

3. Discussion of interactions

In accordance with our earlier remarks concerning the representation of the periodic flow, we let $\psi_{01}(x, y, t)$ be a function of the form

$$\psi_{01}(x,y,t) = \phi_{01}(y)\exp\{i(\beta x-\gamma t)\}+\tilde{\phi}_{01}(y)\exp\{-i(\beta x-\gamma t)\}, \tag{3.1}$$

in which β and γ are taken to be real. We shall later take β to represent the wave-number of the fundamental disturbance, γ the corresponding frequency, and $\phi_{01}(y)$ the eigenfunction associated with β in the linear, parallel flow stability analysis. From (2.15), (2.12) and (3.1), it would seem sufficient to assume that $\psi_{11}(x, y, t)$ is of the form

$$\begin{aligned}\psi_{11}(x,y,t) = {} & \phi_{11,a}(y)\exp[i\{(\alpha_0+\beta)x-(\gamma+\alpha_0 c_0)t\}] \\ & +\phi_{11,b}(y)\exp[i\{(\alpha_0-\beta)x+(\gamma-\alpha_0 c_0)t\}]+\text{conjugate terms}. \end{aligned} \tag{3.2}$$

In general, $\psi_{11}(x, y, t)$ will grow or decay with time in the same manner as the eigensolution $\psi_{10}(x, y, t)$ associated with the mean flow. Although small-amplitude waves of $O(\delta)$ and of wave-number $\alpha_0 \pm \beta$ will therefore appear, the predominant wave will still have the wave-number of the most unstable wave as predicted by the parallel flow analysis. Attention will therefore be restricted to the cases for which the form of (3.2) becomes invalid. Such cases arise when the exponential factors $\{\alpha_0 \pm \beta, \alpha_0 c_0 \pm \gamma\}$ are eigenvalues associated with the linear, mean stability problem, i.e. the homogeneous part of (2.15). A solution would then be possible only if the non-homogeneous terms were orthogonal to the solution of the homogeneous adjoint equation (cf. Ince 1956, §9.34). Because this condition will not be met in general, the form of (3.2) must be revised. One could satisfy the orthogonality condition by assuming that $\psi_{11}(x, y, t)$ has a 'secular' behaviour with time or distance. Such behaviour could be incorporated into the above solution by redefining $\phi_{11,j}(y)$ so that

$$\phi_{11,j}(x,y,t) = \phi^{(1)}_{11,j}(y)+x\phi^{(2)}_{11,j}(y)+t\phi^{(3)}_{11,j}(y)_{j=a,b}. \tag{3.3}$$

The last two functions on the right-hand side of (3.3) would be chosen so that the orthogonality condition is satisfied. Assuming that such a solution is possible it would indicate that the growth rate of the disturbance is altered considerably. We shall later follow a different procedure which allows this feature to be reflected in the growth rate of our zero-order solutions (2.12).

The physical process consists of the periodic flow interacting with waves associated with the mean flow so as to produce waves of the same wave-number and frequency as the latter, which therefore tend to be reinforced. The possibility of resonance thereby arises.

The mechanism is somewhat similar to that discussed by Raetz (1959; cf. Stuart 1962, for further discussion) in connexion with the instabilities which occur in boundary-layer flow on a flat plate and by a number of authors in connexion with non-linear gravity wave interaction (cf. Ball 1964, for the many references concerning this problem). Raetz pointed out that the interaction gives rise to secular growth of the resonant disturbance. Benny & Niell (1962) have shown, however, that the interaction is equivalent to an energy-sharing mechanism between the various modes. In our special case of interest, the data shown in figure 1 indicates that the energy of the fundamental remains almost constant while the subharmonic emerges. This fact suggests that energy is not being shared, but that the periodic flow associated with the fundamental permits further energy to be extracted from the mean flow and fed into the subharmonic disturbance. Our theory differs from that of Raetz because he considered the interaction between growing disturbances, each of which was an eigensolution to the linear, parallel flow stability problem. Here the interaction takes place between the periodic flow, assumed to have arisen from finite-amplitude growth of the fundamental disturbance, and other disturbances which grow rather slowly with time, relative to the initial growth of the fundamental.

Let us consider now the conditions under which the exponential factors in (3.2) can be eigenvalues of the homogeneous part of equation (2.15). To do this, we let (α_0^*, c_0^*) denote any possible pair of eigenvalues for the linear, parallel flow stability problem. Then, from (3.2), the conditions can be listed as

$$\beta = \pm(\alpha_0^* - \alpha_0), \quad \gamma = \pm(\alpha_0^* c_0^* - \alpha_0 c_0), \tag{3.4}$$

$$\beta = \pm(\alpha_0^* + \alpha_0), \quad \gamma = \pm(\alpha_0^* c_0^* + \alpha_0 \tilde{c}_0), \tag{3.5}$$

in which the $(+)$ or $(-)$ sign occurs simultaneously with both β and γ. Relations conjugate to (3.4) and (3.5) are obtained when the conjugate solution (α_0^*, c_0^*) is considered.

It is simplest to discuss two possible cases which arise depending upon whether we take $\alpha_0 = \alpha_0^*$ or $\alpha_0 \neq \alpha_0^*$. Consider first the former case.

(3*A*) $\alpha_0 = \alpha_0^*$, $c_0 = c_0^*$

In this case, we investigate how the periodicity, with wave-number β, interacts with a wave of wave-number α_0 to produce a wave of the same wave-number, α_0. Condition (3.4) is not of interest here because it essentially concerns the problem of how the stability characteristics of a given parallel flow, $\psi_{00}(y)$, are affected by the superposition of a second parallel flow, $\psi_{01}(y)$. While the methods described later can be applied to this problem, it is clear that no insight into novel physical phenomena would result.

Condition (3.5) is, however, quite different. Not only is γ real, as required by our model, but the wavelength of the periodic flow is exactly half of the wavelength of the wave undergoing resonance (we now use this word for convenience; it does not necessarily imply an increase in growth rate, as later results will indicate). For the special case of an antisymmetric shear flow, the phase velocity is independent of wave-number. If the phase velocity is non-zero, the frequency of the resonant wave would be half that of the periodicity, as we expected on the basis of Sato's results.

The stability diagram for the flow (2.13) is shown in figure 2. The diagram is based upon the calculations of Michalke (1964). The fundamental disturbance has a wave-number of 0·4446, which we shall later equate to β. Thus, the resonant wave would have a wave-number of 0·2223.

(3B) $\alpha_0 \neq \alpha_0^*$

We now consider the possibility of the periodicity interacting with a disturbance of wave-number α_0 to produce a wave of wave-number α_0^* equal to the wave-number of some other growing disturbance.

This case is more dependent upon the particular flow considered, and we shall discuss it with reference to figure 2. Condition (3.5) is then not applicable when β corresponds to the wave-number for maximum growth, for we would have to

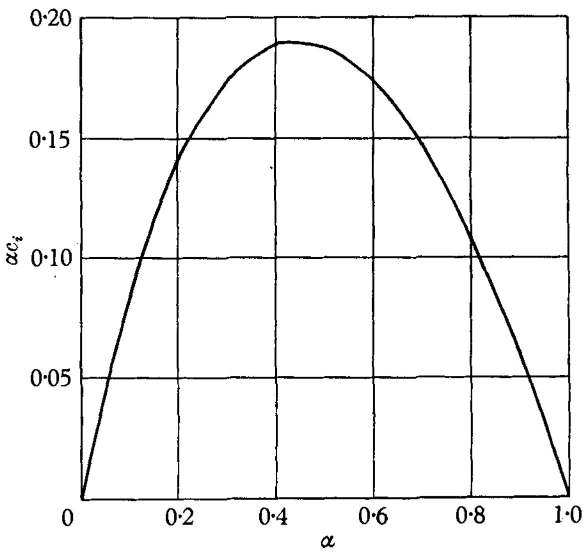

FIGURE 2. Temporal growth rate of disturbances to the flow $U_0(y) = \tanh y$. (From Michalke 1964.)

take both α_0 and α_0^* less than β, thus ensuring that γ would be complex. This would violate the conditions of our model. On the other hand, condition (3.4) would seem to be of importance for our particular flow. In order that γ be real, we must have

$$\alpha_0^* c_{0,i}^* = \alpha_0 c_{0,i}. \tag{3.6}$$

This requires that we take α_0 greater than β and α_0^* less than β, or vice versa. In fact, for the flow with a hyperbolic-tangent profile, the conditions are approximately satisfied for

$$\alpha_0^* = 0{\cdot}234, \quad \alpha_0 = 0{\cdot}679, \tag{3.7}$$

or vice versa. The distinctive feature of this case is therefore that the periodicity might promote simultaneously resonance in two different waves. One wave would be of greater wavelength than the basic periodic flow, whereas the second would be of smaller wavelength.

For convenience, we shall refer to the above cases as A and B, respectively. It is interesting to note that the lower value of wave-number in case B differs from the value given in case A by only 5 %. We shall therefore speculate on the possibility that this difference might be smoothed out in reality and consider the consequences of both of these interactions occurring for the same lower value of wave-number. We shall refer to this case as 'case C'.

The various interactions are shown diagrammatically in figure 3. Later results will show that case A is the most important, at least as far as disturbances which grow with time are concerned.

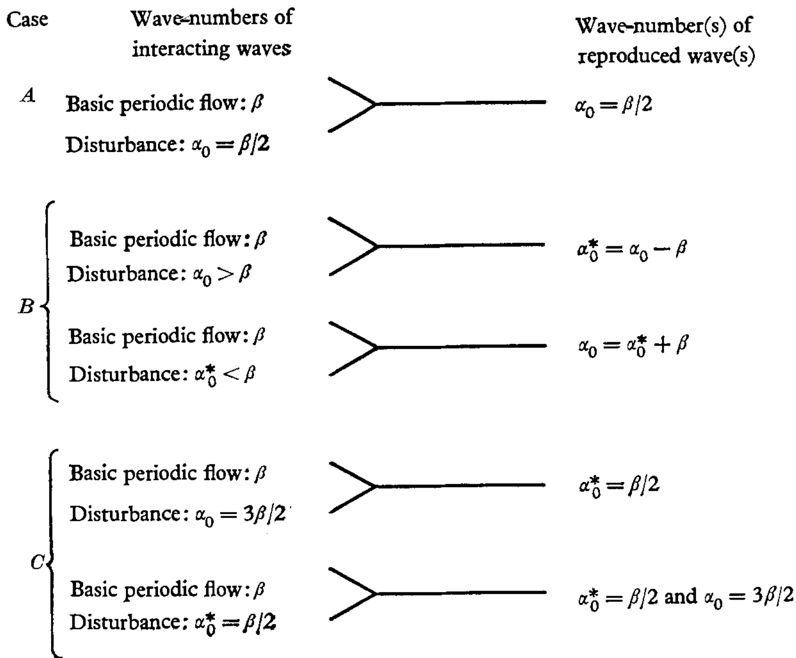

FIGURE 3. Interactions which might cause resonance.

The case B interaction resembles the type of resonance phenomenon discussed in previous investigations concerning non-linear wave interaction (e.g. Raetz 1959), whereas the resonance described under case A has greater resemblance to parametric amplification phenomena. For an example of parametric resonance in the case of an unsteady but parallel Kelvin–Helmholtz flow, the reader is referred to a previous paper by the author (1965). In that paper, emphasis was placed upon the importance of the interaction between the flow oscillations and the dispersive nature of the waves. Because a group of non-dispersive waves can always be regarded as being steady by means of a simple translation of axis, the introduction of another frequency (namely that of the flow oscillation) should not be expected to affect them significantly. In the present paper, however, the basic flow is considered to vary both with time and distance in the flow direction. The chances for the occurrence of the duplication mechanism are much greater when non-dispersive waves are involved, simply because the condition on frequency is rather unimportant and can be easily met. Indeed, it would seem unlikely that case A could occur for flows with dispersive waves. For instance, when we consider the stability of a full jet flow (Lessen & Fox 1955), we find that the phase velocity of the most strongly amplified wave is approximately 50 % greater than that of a wave with half the wave-number. This difference precludes the mechanism discussed in case A from being operative (the case B interaction also does not arise for this flow).

It is perhaps appropriate to comment here briefly on the possible interaction of two growing disturbances of different wave-number so as to cause a third wave to resonate. This is exactly the mechanism discussed by Raetz (1959). With reference to figure 2, it is clear that any wave with wave-number less than that for maximum growth (i.e. $\alpha < 0{\cdot}4446$) can interact with a wave of sufficiently small wave-number so as almost to reproduce itself. Similarly, a wave with wave-number near $\alpha = 0{\cdot}5$ can interact with a wave sufficiently near the neutral disturbance so as almost to reproduce itself. Computations, based upon figure 2, indicate that these are the only possibilities. In either case, the interaction would involve one wave which we can reasonably expect to be of quite small amplitude. This type of interaction should therefore be unimportant for the present problem.

We have also discussed only how the periodic flow might affect the behaviour of growing disturbances associated with the mean flow. Interaction might also occur for decaying disturbances in certain flows, but the effect should be unimportant except when the amplitude of the basic periodic flow is quite large. At any rate, such interaction is irrelevant to the specific flow discussed in this paper. Hence, we omit discussion of the interaction with waves whose wave-numbers are greater than the neutral value or with the low-wave-number, damped waves discussed by Tatsumi, Gotoh & Ayukawa (1964).

4. Analysis of case *A*

We anticipate that the departure of the growth rate of the resonant disturbance from the value predicted by the parallel flow analysis will be of $O(\delta)$ and therefore introduce the additional variables

$$\hat{x} = \delta x, \quad \hat{t} = \delta t, \tag{4.1}$$

to reflect this change. Thus, we shall employ the technique of multiple scaling, as discussed with regard to ordinary differential equations by Cole & Kevorkian (1963). We now define

$$\psi_1 = \psi_1(x, \hat{x}, t, \hat{t}, y). \tag{4.2}$$

With reference to (2.4), the equation for ψ_1 is now

$$\left\{\frac{\partial}{\partial t} + \delta\frac{\partial}{\partial \hat{t}} + \frac{\partial\psi_0}{\partial y}\left(\frac{\partial}{\partial x} + \delta\frac{\partial}{\partial \hat{x}}\right) - \frac{\partial\psi_0}{\partial x}\frac{\partial}{\partial y}\right\}\left\{\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} + 2\delta\frac{\partial^2}{\partial x\,\partial\hat{x}} + \delta^2\frac{\partial^2}{\partial\hat{x}^2}\right\}\psi_1$$
$$+\left(\frac{\partial}{\partial x}\nabla^2\psi_0\right)\frac{\partial\psi_1}{\partial y} - \left(\frac{\partial}{\partial y}\nabla^2\psi_0\right)\left(\frac{\partial\psi_1}{\partial x} + \delta\frac{\partial\psi_1}{\partial\hat{x}}\right) = 0. \tag{4.3}$$

We now expand as

$$\psi_1 = B(\hat{x},\hat{t})\,\phi_{10}(y)\exp\{i\alpha(x-ct)\} + \tilde{B}(\hat{x},\hat{t})\,\tilde{\phi}_{10}(y)\exp\{-i\alpha(x-\tilde{c}t)\}$$
$$+\,\delta\psi_{11}(x,\hat{x},t,\hat{t},y) + \ldots, \tag{4.4}$$

which we denote as

$$\psi_1 = B\phi_{10}E + \tilde{B}\tilde{\phi}_{10}\tilde{E} + \delta\psi_{11} + \ldots. \tag{4.5}$$

Thus, the zero-order solution consists of any eigensolution associated with the mean flow but multiplied by slowly varying functions of $\hat{x}$ and $\hat{t}$. These functions will be determined by ensuring that the previously mentioned orthogonality condition is satisfied for the special case of resonance.

We have here dropped the subscripts on the eigenvalues because we wish to investigate to some extent how the basic periodic flow affects waves with wave-numbers close but not exactly equal to that of the resonant wave. Thus, we now take our conditions as

$$\left.\begin{aligned} \beta = 2\alpha_0, \quad \gamma = 2\alpha_0 c_{0,r}, \quad \alpha &= \alpha_0+\alpha_1, \quad |\alpha_1| \ll \alpha_0, \\ c &= c_0+c_1, \quad |c_1| \sim O|\alpha_1|. \end{aligned}\right\} \tag{4.6}$$

Let us first briefly digress from our argument and consider how the eigensolutions of the mean flow for given (α, c) can be related to the eigensolutions for (α_0, c_0) when the above conditions are applicable. We could expand in terms of α_1 as

$$\phi_{10}(y) = \phi_{10,0}(y)+\alpha_1\phi_{10,1}(y)+\ldots, \tag{4.7}$$

where (α_0, c_0) are the eigenvalues corresponding to $\phi_{10,0}(y)$. The relation between α_1 and c_1 would then be determined while solving for $\phi_{10,1}(y)$. In the special case, however, when $\phi_{10,0}(y)$ corresponds to the neutral disturbance, the desired relation has been given by Lin (1955, §8.2) and is

$$\left(\frac{\partial c}{\partial \alpha^2}\right)_{\alpha=\alpha_s} = -\left\{\int_{-\infty}^{\infty} \phi_s^2 dy\right\} \Big/ \left\{P\int_{-\infty}^{\infty} \frac{\psi_{00}''' \phi_s^2}{(\psi_{00}'-\psi_{00}'(y_s))^2} dy + i\pi \left(\frac{\psi_{00}^{\text{iv}}\phi_s^2}{\psi_{00}''^2}\right)_{y=y_s}\right\}, \tag{4.8}$$

where y_s is the position of the inflexion point, α_s is the neutral wave-number, and P denotes the principal value of the integral.

For the particular case

$$\psi_{00}' = \tanh y, \quad \alpha_s = 1{\cdot}0, \quad \phi_s = \operatorname{sech} y, \quad c_r = 0, \tag{4.9}$$

the above relation gives $\qquad c = -i(\alpha^2-1)/\pi. \qquad (4.10)$

(The result given in §7 of the paper by Drazin & Howard (1962) is in error by a factor of two.) If we now take $\alpha = 1+\alpha_1$, $|\alpha_1| \ll 1$, we have

$$c_1 = -2i\alpha_1/\pi \tag{4.11}$$

to $O(\alpha_1)$. For future reference, we write out the zero-order terms in the expansion (4.4) for the special case (4.9)

$$\begin{aligned} \psi_{10} = {} & B(\hat{x},\hat{t})\{\operatorname{sech} y+\alpha_1\phi_{10,1}\}\exp\{i(1+\alpha_1)x-(2\alpha_1 t/\pi)\} \\ & +\tilde{B}(\hat{x},\hat{t})\{\operatorname{sech} y+\alpha_1\tilde{\phi}_{10,1}\}\exp\{-i(1+\alpha_1)x-(2\alpha_1 t/\pi)\}. \end{aligned} \tag{4.12}$$

The function $\phi_{10,1}(y)$ will not be required for the following analysis. In principle, results analogous to (4.12) can be obtained when $\alpha_0 < 1{\cdot}0$.

Returning now to (4.3) and (4.5), the equation for $\psi_{11}(x,\hat{x},t,\hat{t},y)$ can be written as

$$\begin{aligned} & \left\{\frac{\partial}{\partial t}+\psi_{00}'\frac{\partial}{\partial x}\right\}\nabla^2\psi_{11}-\psi_{00}'''\frac{\partial\psi_{11}}{\partial x} \\ & = -\frac{\partial B}{\partial \hat{t}}(\phi_{10}''-\alpha^2\phi_{10})E+\frac{\partial B}{\partial \hat{x}}\{2\alpha^2(\psi_{00}'-c)\phi_{10}-c(\phi_{10}''-\alpha^2\phi_{10})\}E \\ & \quad -i\alpha BE(\phi_{10}''-\alpha^2\phi_{10})\frac{\partial\psi_{01}}{\partial y}+BE(\phi_{10}'''-\alpha^2\phi_{10}')\frac{\partial\psi_{01}}{\partial x} \\ & \quad -BE\phi_{10}'\left(\frac{\partial}{\partial x}\nabla^2\psi_{01}\right)+i\alpha BE\phi_{10}\left(\frac{\partial}{\partial y}\nabla^2\psi_{01}\right) \\ & \quad + \text{ conjugate terms involving } (\tilde{B},\tilde{E},\tilde{\phi}_{10}). \end{aligned} \tag{4.13}$$

We take $\psi_{01}(x,y,t)$ to be of the form (3.1). Now if $\alpha_1 \sim O(\delta)$, we can let $\alpha \to \alpha_0$, $c \to c_0$, and $\phi_{10} \to \phi_{10,0}$ in the right-hand side of (4.13). Further, we have

$$E \exp\{-i(\beta x - \gamma t)\} = (\tilde{E}) \exp\{2i\alpha_1 x - 2i(\alpha_1 c_{0,r} + \alpha_0 c_{1,r})t\}. \tag{4.14}$$

Thus, $E \exp\{-i(\beta x - \gamma t)\}$ will differ significantly from $\tilde{E}$ only over, say, a length scale large compared to α_0^{-1}. We denote this fact by writing

$$E \exp\{-i(\beta x - \gamma t)\} = (\tilde{E})(1 + O|\delta|). \tag{4.15}$$

We shall restrict the magnitude of α_1 by the condition that $\alpha_1 \sim O|\delta|$ in the following work.

For the conditions given in (4.6), we can then take the solution to (4.13) to be of the form

$$\begin{aligned}\psi_{11} &= \phi_{11,b}(\hat{x},\hat{t},y)\,E + \tilde{\phi}_{11,b}(\hat{x},\hat{t},y)\,\tilde{E} \\ &\quad + \phi_{11,a}(\hat{x},\hat{t},y)\,E \exp\{i(\beta x - \gamma t)\} + \tilde{\phi}_{11,a}(\hat{x},\hat{t},y)\,\tilde{E} \exp\{-i(\beta x - \gamma t)\}.\end{aligned} \tag{4.16}$$

The exponential factors of the last two terms are assumed not to be eigenvalues of the homogeneous equation for case A. Hence $\phi_{11,a}$ and $\tilde{\phi}_{11,a}$ might be obtained without any essential difficulty arising. The equation for $\phi_{11,b}(\hat{x},\hat{t},y)$ is, to $O(\delta)$

$$\begin{aligned} & i\alpha_0(\psi'_{00} - c_0)(\phi''_{11,b} - \alpha_0^2 \phi_{11,b}) - i\alpha_0 \psi'''_{00} \phi_{11,b} \\ &= -(\partial B/\partial \hat{t})(\phi''_{10} - \alpha_0^2 \phi_{10}) + (\partial B/\partial \hat{x})\{2\alpha_0^2(\psi'_{00} - c_0)\phi_{10} - c_0(\phi''_{10} - \alpha_0^2\phi_{10})\} \\ &\quad + i\alpha_0 \phi'_{01}(\tilde{\phi}''_{10} - \alpha_0^2 \tilde{\phi}_{10})\tilde{B} + i\beta\phi_{01}(\tilde{\phi}'''_{10} - \alpha_0^2 \tilde{\phi}'_{10})\tilde{B} \\ &\quad - i\beta(\phi''_{01} - \beta^2 \phi_{01})\tilde{\phi}'_{10}\tilde{B} - i\alpha_0(\phi'''_{01} - \beta^2 \phi^{2\prime}_{01})\tilde{\phi}_{10}\tilde{B}. \end{aligned} \tag{4.17}$$

The equation has the homogeneous solution $\phi_{11,b} \sim \phi_{10}(y)$. In order that a solution exists for (4.17), a necessary and sufficient condition is that the right-hand side of (4.17) be orthogonal to the solution $\Phi_{10}(y)$ of the homogeneous adjoint equation (cf. Ince 1956, §9.34), which, for the above case, is

$$\left.\begin{aligned}(\psi'_{00} - c_0)(\Phi''_{10} - \alpha_0^2 \Phi_{10}) + 2\psi''_{00}\Phi'_{10} = 0; \\ \Phi_{10} \to 0 \quad \text{as} \quad y \to \pm\infty.\end{aligned}\right\} \tag{4.18}$$

The solution to (4.18) is

$$\Phi_{10} = \frac{\phi_{10}}{\psi'_{00} - c_0}. \tag{4.19}$$

The necessity of the orthogonality condition can be realized by writing (4.17) in the form

$$L\phi_{11,b} = r(y) \tag{4.17a}$$

and (4.18) in the form

$$\bar{L}\Phi_{10} = 0 \tag{4.18a}$$

then one can easily show that for our boundary conditions

$$\int_{-\infty}^{\infty} \Phi_{10} L\phi_{11,b}\,dy = \int_{-\infty}^{\infty} \phi_{11,b} \bar{L}\Phi_{10}\,dy = 0.$$

We therefore obtain the necessary condition

$$\int_{-\infty}^{\infty} r(y)\,\Phi_{10}(y)\,dy = 0. \tag{4.20}$$

After imposing this condition in (4.17), we obtain the following equation for $B(\hat{x},\hat{t})$

$$Q(\partial B/\partial \hat{t}) + M(\partial B/\partial \hat{x}) - i\tilde{N}\tilde{B} = 0, \tag{4.21}$$

along with its conjugate. We have defined

$$Q = \int_{-\infty}^{\infty} (\phi''_{10} - \alpha_0^2 \phi_{10})\, \Phi_{10} dy = \int_{-\infty}^{\infty} \frac{\psi'''_{00}\phi_{10}\Phi_{10}}{\psi'_{00} - c_0} dy, \tag{4.22}$$

$$M = -2\alpha_0^2 \int_{-\infty}^{\infty} (\psi'_{00} - c_0)\, \phi_{10}\Phi_{10} dy + c_0 Q, \tag{4.23}$$

$$\tilde{N} = \alpha_0 \int_{-\infty}^{\infty} \phi'_{01}(\tilde{\phi}''_{10} - \alpha_0^2 \tilde{\phi}_{10})\, \Phi_{10} dy + \beta \int_{-\infty}^{\infty} \phi_{01}(\tilde{\phi}'''_{01} - \alpha_0^2 \tilde{\phi}'_{10})\, \Phi_{10} dy$$
$$- \beta \int_{-\infty}^{\infty} (\phi''_{01} - \beta^2 \phi_{01}) \tilde{\phi}'_{10}\Phi_{10} dy - \alpha_0 \int_{-\infty}^{\infty} (\phi'''_{01} - \beta^2 \phi'_{01}) \tilde{\phi}_{10}\Phi_{10} dy. \tag{4.24}$$

From (4.21) and its conjugate, we can write the governing equation for $B(\hat{x},\hat{t})$ as

$$\left\{ |Q|^2 \frac{\partial^2}{\partial \hat{t}^2} + 2(Q\tilde{M})_r \frac{\partial^2}{\partial \hat{t}\,\partial \hat{x}} + |M|^2 \frac{\partial^2}{\partial \hat{x}^2} - |N|^2 \right\} B = 0. \tag{4.25}$$

Thus, if we take $B(\hat{x},\hat{t})$ of the form

$$B(\hat{x},\hat{t}) \sim \exp(\lambda \hat{x} + \mu \hat{t}), \tag{4.26}$$

the characteristic equation relating λ and μ is

$$|Q|^2 \mu^2 + 2(Q\tilde{M})_r\, \lambda\mu + |M|^2 \lambda^2 = |N|^2. \tag{4.27}$$

We have here allowed the resonance to be manifested by both spatial and temporal growth of the disturbance, although only the latter behaviour is usually considered directly in theoretical analyses concerning the stability of fluid flows. For real μ and λ, the maximum growth with $\hat{x}$ or $\hat{t}$ may be found by setting μ or λ, respectively, equal to zero. If λ is imaginary, we are effectively changing the wave-number of the disturbance and so expect, from figure 2, the growth rate to change. For the present case we shall look at a disturbance of fixed wave-number and frequency and so take λ and μ to be real.

In general, further progress can be made only by numerical evaluation of the above integrals. Before considering the results of such calculations, it is worth while first to consider a hypothetical case in which the periodic flow causes the neutral disturbance associated with the mean flow (2.13) to resonate. Because analytical determination of Q and M is then possible, we can obtain some idea of how the periodic flow affects waves with wave-numbers close but not exactly equal to the resonant value. It should, of course, be realized that the case is strictly hypothetical, because, with reference to figure 2, we see that a disturbance whose wave-number is twice the neutral value tends to be strongly damped. The analysis is intended only for the purpose of discussion.

For the flow (2.13), the neutral solution is described by (4.9), and the adjoint solution is

$$\Phi_{10}(y) = \operatorname{cosech} y \tag{4.28}$$

which is singular as $y \to 0$. In order that finite values of Q and M can be obtained, a viscous correction to $\Phi_{10}(y)$ must be made, or, alternatively, the integrations may be made below the critical point in the complex plane, thus ensuring that we take the limit $c_i \to 0$ from above (cf. Lin 1955, §8.2, §8.3). Schade (1964) has given the details of the first approach for the present flow. He has shown that the adjoint solution, with the viscous correction, is

$$\Phi_{10,\,\mathrm{corr}} = iYL(Y)\,\mathrm{cosech}\,y \tag{4.29}$$

where $Y = y/\sigma$, σ being a small parameter equal to $(\alpha R)^{-\frac{1}{3}}$ and R the Reynolds number. The function $L(Y)$ is the Lommel function, whose real and imaginary parts behave as

$$L_r \sim -2/Y^4, \quad L_i \sim -1/Y, \tag{4.30}$$

for $Y \to \infty$. We see that (4.29) tends to (4.28) in this limit. On the other hand, $L(0)$ is finite, and so (4.29) is finite as $y \to 0$.

We can now evaluate (4.22) through the use of (4.29) and the knowledge that $L_r(Y)$ is even while $L_i(Y)$ is odd.

$$\begin{aligned} Q &= -2\int_{-\infty}^{\infty} \frac{iYL(Y)}{\cosh^3 y \sinh y}\,dy = -2\lim_{\sigma Y \to 0}\int_{-\infty}^{\infty} \frac{i\sigma YL(Y)\,dY}{\cosh^3(\sigma Y)\sinh(\sigma Y)} \\ &= -2i\int_{-\infty}^{\infty} L_r(Y)\,dY = -2i\pi. \end{aligned} \tag{4.31}$$

We also have

$$M = -2\int_{-\infty}^{\infty} \mathrm{sech}^2 y\,dy = -4. \tag{4.32}$$

The characteristic relation is then

$$4\pi^2\mu^2 + 16\lambda^2 = |N|^2. \tag{4.33}$$

This result indicates that although the disturbance might grow both spatially and temporally it must at least grow either with time or with distance in the direction of flow. It is customary to discuss disturbances which grow with time. If we take $\lambda = 0$, we can state with reference to (4.12) that ψ_{10} behaves as

$$\psi_{10} \sim (\mathrm{sech}\,y + \alpha_1\phi_{10,1}(y))\exp\left\{i(1+\alpha_1)x + \left[-\frac{2\alpha_1}{\pi} \pm \frac{|N|\delta}{2\pi}\right]t\right\}. \tag{4.34}$$

The choice of signs reflects essentially the arbitrariness of the sign of δ. Thus, there are two solutions but one will always grow faster than the other. On the basis of (4.34), we can say that the neutral wave-number is shifted in this hypothetical case so that

$$\alpha_s = 1 + \tfrac{1}{4}|N\delta| \tag{4.35}$$

to $O(\delta)$. Thus, the periodic flow considered is unstable with respect to disturbances with a wider range of wave-number than is the associated mean flow.

We also see from (4.34) that slightly unstable waves ($\alpha_1 < 0$) will have their growth controlled to a greater extent by the instability connected with the mean flow as $|\alpha_1|$ increases. In the rest of this work, we shall consider only waves which meet exactly the conditions for resonance. However, it should be remembered that the above analysis indicates that waves which almost meet the requirements for resonance can also be destabilized.

Having established this point, we now consider cases of greater interest that require the numerical evaluation of (4.22)–(4.24). In order to consider the effects of the finite growth of the primary instability of the flow (2.13), we take $\beta = 0{\cdot}4446$, $\alpha_0 = 0{\cdot}2223$, and $\phi_{01}(y)$ to be the eigenfunction associated with the most unstable disturbance in the parallel flow analysis. Thus, the effect of finite-amplitude growth on the shape of the fundamental disturbance is neglected. With reference to (2.9), we expect that such distortion should be of $O(\delta^3)$, which is of higher order than the effect of such growth on the mean flow.

The numerical calculations were generously performed for the author by Prof. W. C. Reynolds on an IBM 7094. The eigenfunctions were determined by inward numerical integration of the linear stability equation, using the eigenvalues given by Michalke (1964). The equation was reduced to a pair of first-order differential equations, and a four-point Adams predictor–corrector scheme was employed. The integration began at $y = 8{\cdot}0$ with the known asymptotic form of the solutions and used 400 steps. The eigenfunctions, which are composed of real symmetric and imaginary antisymmetric components, were normalized so that

$$\phi_r(0) = 1, \quad \phi_i(0) = 0 \tag{4.36}$$

so as to agree with the normalization used by Michalke (1964). The eigenfunctions are shown in figure 5 of his paper. The integrals were then calculated by means of Simpson's rule and the known asymptotic behaviour of the various functions for large y. The program was written in Fortran IV, using the automatic complex arithmetic provisions.

	α_0	β	Q	M	N
(1)	0·2223	0·4446	$-1{\cdot}2312i$	0·4813	$-0{\cdot}3302$
(2)	0·2223	0·4446	$-1{\cdot}2312i$	0·4813	$-0{\cdot}3111$
(3)	0·50	1·0	$-3{\cdot}5594i$	$-0{\cdot}2800$	$-3{\cdot}4342$

	$\lambda\vert_{\mu=0}$	$\mu\vert_{\lambda=0}$
(1)	$\pm 0{\cdot}6860$	$\pm 0{\cdot}2682$
(2)	$\pm 0{\cdot}6464$	$\pm 0{\cdot}2527$
(3)	$\pm 12{\cdot}26$	$\pm 0{\cdot}9648$

TABLE 1

The results are given in table 1 on line (1). Let us again consider only disturbances which can grow with time, so that

$$\psi_{10} \sim \phi_{10}(y) \exp\{i\alpha_0 x + (\alpha_0 c_{0,i} \pm \mu\delta)t\}. \tag{4.37}$$

From figure 2, we see that $\alpha_0 c_{0,i} \simeq 0{\cdot}15$ for $\alpha_0 \simeq 0{\cdot}22$, whereas the maximum growth rate is approximately 0·19. Using the calculated value of μ, we find that the total growth rate for the subharmonic wave is approximately equal to the maximum growth rate predicted by the parallel flow analysis when $\delta \simeq 0{\cdot}15$. For values of δ below this value, we can say that the most unstable wave associated with the periodic flow has the same wave-number as that associated with the mean flow. A plausible conclusion is that the periodic flow will remain distinguished for the most part by the wave-number $\beta = 0{\cdot}4446$ for $\delta < 0{\cdot}15$. For $\delta > 0{\cdot}15$, however, the subharmonic wave will be the predominant instability associated with the periodic flow. The flow will then be mainly described by the two wave-numbers $\beta = 0{\cdot}4446$ and $\alpha_0 = 0{\cdot}2223$.

We shall now use this threshold value of δ in order to estimate the amplitude required for the periodic flow to make the growth rate of the subharmonic wave predominant. The basic flow can be expressed as

$$U_0 = \frac{\partial \psi_0}{\partial y} = \tanh y + \delta \left(\frac{\partial \phi_{01}}{\partial y} e^{i\beta x} + \frac{\partial \tilde{\phi}_{01}}{\partial y} e^{-i\beta x} \right) \tag{4.38}$$

where $\beta = 0{\cdot}4446$. From figure 5 of Michalke's (1964) paper, we estimate that the maximum value of $|\partial\phi_{01}/\partial y|$ is about 0·8. Thus, a value $\delta = 0{\cdot}15$ implies that the amplitude of the fundamental disturbance must be approximately 12 % of the velocity difference across the shear layer. This estimate is above Sato's (1959, figure 17) measurement of the root-mean-squared value of the disturbance (3–10 %) in the non-linear region upstream of the subharmonic region and below Schade's (1964) theoretical estimate of the possible equilibrium amplitude of the fundamental disturbance (17 %).

In order to compare the estimated value of δ with experiment, we should first apply our results to the flow

$$U_0 = 0{\cdot}5(1 + \tanh y) \tag{2.14}$$

and then consider disturbances which grow spatially in the downstream direction. The growth rates of disturbances to the flow (2.14) which grow with time are half of the values given in figure 2. We also now have $c_{0,r} = 0{\cdot}5$. By inspection of (4.22), (4.24) and (4.27), we see that the value of μ for $\lambda = 0$ is equal to that given in table 1. Hence, the threshold value of δ is 0·075, and the corresponding amplitude of the streamwise component of the periodic flow remains about 12 % of the mean velocity difference, as far as disturbances which grow with time are concerned.

The relation of the present results to disturbances which grow spatially is not so straightforward for several reasons. First, while the possibility of spatial growth has been included in our analysis for the sake of generality, the actual estimation of the spatial growth factor (λ in table 1) is unreliable because it was made by use of the eigenfunctions associated with temporally growing disturbances. If we do take this value, scale it so as to be appropriate to the flow (2.14), and match the spatial growth rate of the subharmonic to that of the maximum permissible as predicted by a parallel flow analysis (see appendix), we find that the critical value of δ is about 0·10. It is, however, doubtful whether the spatially growing disturbances associated with the flow (2.14) can even be involved in the mechanism of resonance which was discussed in §3. As shown by Michalke (1965*b*) and in the appendix to this paper, the spatially growing disturbances show a distinct variation of phase velocity with wave-number, so that the frequency for waves with $\alpha_{0,r}$ less than 0·4446 can be greater than $0{\cdot}5\alpha_{0,r}$ by as much as 40 %. For waves with $\alpha_{0,r}$ greater than 0·4446, the frequency is somewhat less than $0{\cdot}5\alpha_{0,r}$. This dispersive character of the spatially growing solutions means that the conditions for resonance, as set forth in §3, are unlikely to be satisfied.

On the other hand, there seems to be little reason to jettison the mechanism of resonance discussed above, for it does point to the existence of a subharmonic oscillation, has appeal on a physical basis, and does predict a critical amplitude for the primary disturbance which agrees reasonably well with experiment. One possible path out of the present difficulties lies in noting that, as shown by Michalke (1964, 1965*b*), the wave-number and frequency of the disturbance which undergoes maximum spatial growth differs from that which undergoes maximum temporal growth by only 10 %. We have already shown that waves which almost satisfy the conditions for resonance can be destabilized. Hence, it is plausible that the flow associated with the primary spatial disturbance can promote a subharmonic oscillation which would then grow temporally with a growth rate proportional to the amplitude of the primary disturbance. Further calculations, in which the periodic flow is chosen to represent the velocity distribution associated with the fundamental spatial instability, would be required to judge the validity of this conjecture.

To return now to our previous results, it should be noted that they are valid, of course, only to terms of $O(\delta)$. If terms of $O(\delta^2)$ were of significance, other interactions might become important, e.g. those with $\alpha_0 = \beta$ or $\alpha_0 = 3\beta/2$. The situation is similar to those involving parametric resonance phenomena, which are usually described by a differential equation of the Mathieu type (cf. Stoker 1950, chap. VI). In such situations, there are an infinite number of possibilities for resonance; the subharmonic case, however, is usually the most important.

Before leaving this case, we consider first two other results concerning the flow (2.13) and which are given in table 1. On the second line are the values obtained when $\alpha_0 = 0{\cdot}2223$ and $\beta = 0{\cdot}4446$ but when $\phi_{01}(y)$ is the eigenfunction associated with the neutral solution, i.e. $\phi_{01}(y) = \operatorname{sech} y$. This computation was done partly because Sato (1959, figure 15) measured a profile in the subharmonic region which resembled the neutral disturbance profile and partly because a standard for comparison was desired. As is evident, the actual value of μ is only slightly less than in the previous case. Because the maximum value of $|\partial\phi_{01}/\partial y|$ is only about 0·5 for this case, the threshold value of the amplitude of the streamwise component of the periodic flow is only about 8 % of the mean velocity difference across the shear layer. The result indicates that the actual vorticity distribution associated with the periodic flow may not be of primary importance; the important factors are felt to be the periodic character of the flow and the non-dispersive nature of the waves associated with the basic parallel flow.

On the third line of table 1, the results are given for $\alpha_0 = 0{\cdot}5$, $\beta = 1{\cdot}0$, and $\phi_{01}(y) = \operatorname{sech} y$. The result is of interest because it concerns the stability of the flow described by the exact solution (2.8) to the equations of motion in the limit $C^2 \to 1+\delta^2$. The periodic flow tends to reinforce the wave with $\alpha_0 = 0{\cdot}5$. The factor μ is almost four times the value given in the previous calculations, and indicates that quite small values of δ can shift the most unstable wave-numbers from 0·4446 to 0·5. While we might be tempted to regard this result as an indication of how the neutral disturbance associated with the flow (2.13) interacts with the wave with $\alpha_0 = 0{\cdot}5$, we should remember that the amplitude of the neutral disturbance must also be allowed to vary during such interaction. In this paper, we have regarded the periodic flow as being of fixed amplitude because this seems to be true for our particular application (cf. § 1).

5. Analysis of case B

As mentioned earlier, the results for cases B and C indicate that these interactions have less importance for the growth of the subharmonic wave. However, they are of interest because they indicate that satisfaction of the conditions of 'resonance', as set forth in § 3, does not automatically imply increased temporal growth for the interacting waves.

With reference to the discussion of § 3 B, we now consider the interaction of the basic periodic flow, of wave-number β, with a wave of wave-number α_0 so as to produce a wave with wave-number $\alpha_0^* \neq \alpha_0$, and vice versa. Both α_0 and α^* are assumed to be possible eigenvalues of the parallel flow stability problem. Using the notation defined in (4.5) ,we write $\psi_1(x, \hat{x}, t, \hat{t}, y)$ as

$$\psi_1 = A\phi_{10}E + \tilde{A}\tilde{\phi}_{10}\tilde{E} + B\phi_{10}^* E^* + \tilde{B}\tilde{\phi}_{10}^* \tilde{E}^* + \delta\psi_{11}, \tag{5.1}$$

for which the definition of E^* is obvious.

The basic periodicity is again assumed to be of the form (3.1), but β, α_0 and α_0^* are now related by

$$\beta = \alpha_0 - \alpha_0^*, \quad \alpha_0 c_{0,i} = \alpha_0^* c_{0,i}^*, \quad \gamma = \alpha_0 c_{0,r} - \alpha_0^* c_{0,r}^*. \tag{5.2}$$

For our special case, α_0 and α_0^* are given by (3.7) and $\gamma = 0$. The following identity now holds

$$\exp\{-i(\beta x - \gamma t)\} E = E^*. \tag{5.3}$$

In contrast to (4.16), we now write $\psi_{11}(x, \hat{x}, t, \hat{t}, y)$ as

$$\psi_{11} = \phi_{11,b} E + \phi_{11,b}^* E^* + \phi_{11,a} E \exp\{i(\beta x - \gamma t)\} + \phi_{11,a}^* E^* \exp\{-i(\beta x - \gamma t)\} + \text{conjugate terms.} \tag{5.4}$$

The equations for the various components are obtained by substituting (5.1) and (5.4) into (4.3) and matching terms of $O(\delta)$ with the same exponent. In the present case, it is assumed that only the first two terms of (5.4), together with their conjugates, are involved in resonance. The exponents of the other two terms are considered not to be eigenvalues of the parallel flow stability problem. The equation for $\phi_{11,b}$ is

$$\begin{aligned} &i\alpha_0(\psi_{00}' - c_0)(\phi_{11,b}'' - \alpha_0^2\phi_{11,b}) - i\alpha_0\psi_{00}'''\phi_{11,b} \\ &\quad = -(\partial A/\partial\hat{t})(\phi_{10}'' - \alpha_0^2\phi_{10}) + (\partial A/\partial\hat{x})\{2\alpha_0^2(\psi_{00}' - c_0)\phi_{10} - c_0(\phi_{10}'' - \alpha_0^2\phi_{10})\} \\ &\qquad - i\alpha_0^*\phi_{01}'(\phi_{10}^{*\prime\prime} - \alpha_0^{*2}\phi_{10}^*)B + i\beta\phi_{01}(\phi_{10}^{*\prime\prime\prime} - \alpha_0^{*2}\phi_{10}^{*\prime})B \\ &\qquad - i\beta(\phi_{01}'' - \beta^2\phi_{01})\phi_{10}^{*\prime}B + i\alpha_0^*(\phi_{01}''' - \beta^2\phi_{01}')\phi_{10}^*B. \end{aligned} \tag{5.5}$$

The equation for $\phi_{11,b}^*$ is similar but with the following terms interchanged in turn wherever possible on the right-hand side

$$(\alpha_0, c_0, A, \phi_{10}, \phi_{01}, \beta) \longleftrightarrow (\alpha_0^*, c_0^*, B, \phi_{10}^*, \tilde{\phi}_{01}, -\beta). \tag{5.6}$$

A homogeneous solution of (5.5) which satisfies the homogeneous boundary conditions is $\phi_{11,b} \sim \phi_{10}(y)$. Hence, the right-hand side must be orthogonal to the homogeneous adjoint solution, $\Phi_{10}(y)$, for a solution to exist. A similar statement holds for the equation governing $\phi^*_{11,b}$, except that the non-homogeneous terms must be orthogonal to $\Phi^*_{10}(y)$. The application of these two conditions gives rise to the following equations for $A(\hat{x},\hat{t})$ and $B(\hat{x},\hat{t})$, which may be compared to (4.21):

$$Q\,\partial A/\partial\hat{t}+M\,\partial A/\partial\hat{x}+iRB = 0, \tag{5.7}$$

$$Q^*\,\partial B/\partial\hat{t}+M^*\,\partial B/\partial\hat{x}+iR^*A = 0. \tag{5.8}$$

If the equations for $\tilde{\phi}_{11,b}$ and $\tilde{\phi}^*_{11,b}$ were considered, equations conjugate to (5.7) and (5.8) would be obtained. The functions Q and M are defined as in (4.22) and (4.23) except that now $\alpha_0 = 0{\cdot}679$ in our special case. The functions Q^* and M^* are obtained by letting $(\alpha_0, c_0, \phi_{10}, \Phi_{10}) \to (\alpha^*_0, c^*_0, \phi^*_{10}, \Phi^*_{10})$ in these formulae. The functions R and R^* are given by

$$\begin{aligned} R = {} & \alpha^*_0\int_{-\infty}^{\infty}(\phi^{*''}_{10}-\alpha^{*2}_0\phi^*_{10})\,\phi'_{01}\,\Phi_{10}\,dy-\beta\int_{-\infty}^{\infty}(\phi^{*'''}_{10}-\alpha^{*2}_0\phi^{*'}_{10})\,\phi_{01}\,\Phi_{10}\,dy \\ & +\beta\int_{-\infty}^{\infty}(\phi''_{01}-\beta^2\phi_{01})\,\phi^{*'}_{01}\,\Phi_{10}\,dy-\alpha^*_0\int_{-\infty}^{\infty}(\phi'''_{01}-\beta^2\phi'_{01})\,\phi^*_{10}\,\Phi_{10}\,dy, \end{aligned} \tag{5.9}$$

$$\begin{aligned} R^* = {} & \alpha_0\int_{-\infty}^{\infty}(\phi''_{10}-\alpha^2_0\phi_{10})\,\tilde{\phi}'_{01}\,\Phi^*_{10}\,dy+\beta\int_{-\infty}^{\infty}(\phi'''_{10}-\alpha^2_0\phi'_{10})\,\tilde{\phi}_{01}\,\Phi^*_{10}\,dy \\ & -\beta\int_{-\infty}^{\infty}(\tilde{\phi}''_{01}-\beta^2\tilde{\phi}_{01})\,\phi'_{10}\,\Phi^*_{10}\,dy-\alpha_0\int_{-\infty}^{\infty}(\tilde{\phi}'''_{01}-\beta^2\tilde{\phi}'_{01})\,\phi_{10}\,\Phi^*_{10}\,dy. \end{aligned} \tag{5.10}$$

If $A(\hat{x},\hat{t})$ and $B(\hat{x},\hat{t})$ are assumed to have the same exponential dependence upon $\hat{x}$ and $\hat{t}$ as given in (4.26), the characteristic equation relating λ and μ is

$$QQ^*\mu^2+(QM^*+MQ^*)\,\mu\lambda+MM^*\lambda^2+RR^* = 0. \tag{5.11}$$

α_0	α^*_0	β	Q	Q^*	M
0·679	0·234	0·4446	$-5{\cdot}1164i$	$-1{\cdot}3258i$	$-1{\cdot}5541$
M^*	R	R^*	$\lambda\vert_{\mu=0}$	$\mu\vert_{\lambda=0}$	
0·4866	$-1{\cdot}8099$	0·7002	$\pm 1{\cdot}2946i$	$\pm 0{\cdot}4322i$	

TABLE 2

The various quantities entering into (5.11) are listed in table 2 for our special case when $\phi_{01}(y)$ represents the eigenfunction of the primary disturbance associated with the flow (2.13). If we again consider only disturbances which grow with time and ask how the temporal behaviour of a wave with fixed wave-number (i.e. $\lambda = 0$) is affected, we see that such waves will obtain a non-zero phase velocity but will grow with time as predicted by the parallel flow stability analysis. Alternatively, if we consider a wave of zero phase velocity (i.e. $\mu_i = 0$), the results predict a shift in wave-number but no tendency towards spatial growth. These results are just the opposite of those found in the previous section, and we conclude that the type of interaction described under case B does not result in any secondary instability.

Although a change in the group velocity of a wave which grows with time is often associated with an increased rate of growth for waves which grow spatially (cf. (2.15) for $\delta = 0$), the discussion following (2.15) indicates that spatially growing waves will not enter into resonant interaction.

6. Analysis of case *C*

Finally, we consider the simultaneous occurrence of both of the previous interactions. Thus, the basic periodicity, of wave-number β, is assumed to interact with a wave of wave-number α_0^* so as to reproduce that wave. However, it is assumed to be able to reproduce that wave also through interaction with a wave of wave-number $\alpha_0 \neq \alpha_0^*$. As stated in §3*B*, the conditions for this double interaction are not met exactly for our particular flow, because the quantities $|\beta - \alpha_0^*|$ and $|\alpha_0 - \beta|$ differ by about 5%. Because the difference is rather small and also because we have shown previously that waves close to the resonant wave are affected by the interaction, we shall assume that the difference can be smoothed out and make use of our previous results to analyse this case.

We again take $\psi_1(x, \hat{x}, t, \hat{t}, y)$ of the form (5.1). The following equalities relating α_0, α_0^* and β now hold:

$$\beta = \alpha_0 - \alpha_0^* = 2\alpha_0^*, \quad \gamma = \alpha_0^*(c_0^* + \tilde{c}_0^*) = \alpha_0 c_{0,r} - \alpha_0^* c_{0,r}^*. \tag{6.1}$$

In addition to (5.3), we now have the relation

$$\exp\{-i(\beta x - \gamma t)\} E^* = \tilde{E}^* \tag{6.2}$$

(which is (4.15) with the asterisk notation). In place of (5.4), we now write

$$\psi_{11} = \phi_{11,b} E + \phi_{11,b}^* E^* + \phi_{11,a} E \exp\{i(\beta x - \gamma t)\} + \text{conjugate terms}. \tag{6.3}$$

The equation for $\phi_{11,b}$ is identical to (5.5). In contrast, the equation for $\phi_{11,b}^*$ contains the additional terms which represent the effects of the double interaction and is

$$\begin{aligned}
i\alpha_0^*(\psi_{00}' - c_0^*)(\phi_{11,b}^{*\prime\prime} - \alpha_0^{*2}\phi_{11,b}^*) - i\alpha_0^* \psi_{00}''' \phi_{11,b} \\
= -(\partial B/\partial \hat{t})(\phi_{10}^{*\prime\prime} - \alpha_0^{*2}\phi_{10}^*) + (\partial B/\partial \hat{x})\{2\alpha_0^{*2}(\psi_{00}' - c_0^*)\phi_{10}^* - c_0^*(\phi_{10}^{*\prime\prime} - \alpha_0^{*2}\phi_{10}^*)\} \\
- iA\{\alpha_0(\phi_{10}'' - \alpha_0^2\phi_{10})\tilde{\phi}_{01}' + \beta(\phi_{10}''' - \alpha_0^2\phi_{10}')\tilde{\phi}_{01} \\
- \beta(\tilde{\phi}_{01}'' - \beta^2\tilde{\phi}_{01})\phi_{10}' - \alpha_0(\tilde{\phi}_{01}''' - \beta^2\tilde{\phi}_{01}')\phi_{10}\} \\
+ i\tilde{B}\{\alpha_0^*(\tilde{\phi}_{10}^{*\prime\prime} - \alpha_0^{*2}\tilde{\phi}_{10}^*)\phi_{01}' + \beta(\tilde{\phi}_{10}^{*\prime\prime\prime} - \alpha_0^{*2}\tilde{\phi}_{10}^{*\prime})\phi_{01} \\
- \beta(\phi_{01}'' - \beta^2\phi_{01})\tilde{\phi}_{10}^{*\prime} - \alpha_0^*(\phi_{01}''' - \beta^2\phi_{01}')\tilde{\phi}_{10}^*\}.
\end{aligned} \tag{6.4}$$

After applying the orthogonality condition, we obtain (5.7), along with its conjugate, as well as an equation which results from (6.4) of the form

$$Q^*(\partial B/\partial \hat{t}) + M^*(\partial B/\partial \hat{x}) + iR^*A - i\tilde{N}^*B = 0 \tag{6.5}$$

along with its conjugate. The function $\tilde{N}^*$ is obtained from (4.24) by substituting in turn $(\alpha_0^*, \tilde{\phi}_{10}^*, \Phi_{10}^*)$ for $(\alpha_0, \tilde{\phi}_{10}, \Phi_{10})$.

If we now define the following operators

$$L = Q(\partial/\partial \hat{t}) + M(\partial/\partial \hat{x}), \quad L^* = Q^*(\partial/\partial \hat{t}) + M^*(\partial/\partial \hat{x}) \tag{6.6}$$

we can write the governing equation for $\tilde{B}(\hat{x}, \hat{t})$ as

$$\{(LL^* + RR^*)(\tilde{L}\tilde{L}^* + \tilde{R}\tilde{R}^*) - |N^*|^2 L\tilde{L}\}\tilde{B} = 0. \tag{6.7}$$

For simplicity, we shall consider a disturbance of fixed wave-number which can grow only with time, i.e. $B = B(\hat{t})$. Using the previously defined notation (4.26), the characteristic equation is

$$|Q|^2|Q^*|^2\mu^4+\{2(RR^*QQ^*)_r-|N^*|^2|Q|^2\}\mu^2+|R|^2|R^*|^2 = 0. \tag{6.8}$$

We shall now assume that the various quantities given in tables 1 and 2 may be employed for the calculation of μ in spite of the 5 % difference in eigenvalues. However, in order to have some idea of the errors involved, the calculations have been done twice, first with the value of Q^* corresponding to $\alpha_0^* = 0{\cdot}2223$ (table 1, line 1) and then with the value of Q^* corresponding to $\alpha_0^* = 0{\cdot}234$ (table 2). In each case, four different complex roots were obtained; these are

$$(1)\ \ Q^* = -1{\cdot}2312i:\ \mu = \pm(0{\cdot}1327\pm0{\cdot}4276i), \tag{6.9}$$

$$(2)\ \ Q^* = -1{\cdot}3258i:\ \mu = \pm(0{\cdot}1245\pm0{\cdot}4139i). \tag{6.10}$$

The variations seem small enough that we can assume that the calculations are not nugatory. The real part of μ is only about half of the value calculated for case A, and it would therefore seem that the occurrence of a B-type interaction with an A-type interaction has a strong stabilizing influence upon the latter. Hence we can say that the basic periodic flow in our case exerts its greatest influence when it is able to reproduce a wave by interaction with that wave and with no other.

The greatly reduced value of the real part of μ, relative to case A, means that the streamwise component of the periodic flow would have to be of much larger amplitude (about 24 % of the mean velocity difference) than we could expect on the basis of experiment in order to make the subharmonic growth rate predominant. Of course, further analysis might show that the 5 % difference in the wave-numbers required for the two types of interaction is of greater importance than has been assumed. Also, the small magnitude of the difference might be peculiar to the flow with a hyperbolic-tangent profile and would be increased for flows with still more realistic profiles.

7. Discussion

In this section, we shall summarize the results and then attempt to clarify the main physical process.

We have essentially considered the stability of a flow composed of a mean, parallel component and a non-parallel component which has a periodic behaviour with time and distance in the direction of flow. Our main result is that the periodic flow can destabilize a disturbance associated with the mean flow if the frequency and wave-number of that disturbance are half of those associated with the basic periodic flow. The disturbance and the periodic flow can then interact to produce a wave which is of the same wave-number and frequency as the disturbance and which can therefore reinforce it. Such destabilization seems most likely for non-dispersive waves, such as the temporally growing disturbances possible in antisymmetric shear flows, because the linear dependence of frequency upon wave-number for such waves guarantees that the condition on frequency is met if the condition on wave-number is satisfied.

The process of destabilization was investigated in detail in the case when the mean flow profile was described by the hyperbolic-tangent function. The periodic flow was chosen to represent the finite-amplitude, near-equilibrium flow associated with the fundamental disturbance. A critical amplitude for the periodic flow was defined by finding the amplitude required to make the subharmonic growth rate exceed that of the most unstable disturbance associated with the mean flow. The critical amplitude of the streamwise component of the periodic flow was found to be about 12 % of the mean velocity difference across the shear layer. This is above the value (3–10 %) which was expected on the basis of an experiment performed by Sato (1959) in connexion with a separated shear flow, but the order-of-magnitude agreement seems acceptable.

A second mechanism by which the periodic flow might affect the behaviour of a wave with wave-number less than its own was revealed during the analysis. This interaction takes place between the periodic flow and two different waves, whose difference in wave-number and frequency is equal to those of the periodic flow. The numerical results indicate that this type of interaction does not result in increased amplification with time of the interacting waves.

Some interpretation of the above results is obviously desirable. The concept of reinforcement of a wave through wave reproduction seems clear enough. The special propensity of the subharmonic wave to be reproduced, however, needs clarification, as well as the failure of the 'double' interaction mechanism to result in reinforcement.

Some insight can be gained by examining the energy of that portion of the flow which is described by the subharmonic wave-number (say α) and odd multiples of it (3α, 5α, etc.). Let this flow be denoted by u', v'. The rest of the flow can be split into a mean component, $\overline{u}$, and a portion described by even multiples of the subharmonic wave-number (2α, 4α, etc.), which includes the wave-number (2α) of the basic periodic flow. Let the flow described by these wave-numbers be denoted by u'', v''. Stuart (1962, equation 4.9) has derived the energy equation for the u', v' flow, which is

$$\frac{\partial}{\partial t}\int\int \tfrac{1}{2}(u'^2+v'^2)\,dx\,dy = \int\int(-u'v')\frac{\partial \overline{u}}{\partial y}\,dx\,dy - \frac{1}{R}\int\int\left(\frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y}\right)^2 dx\,dy$$
$$-\int\int\left[(u'^2-v'^2)\frac{\partial u''}{\partial x}+u'v'\left(\frac{\partial u''}{\partial y}+\frac{\partial v''}{\partial x}\right)\right]dx\,dy, \qquad (7.1)$$

where the integration takes place over the wavelength ($2\pi/\alpha$) of the subharmonic disturbance and the flow width. The first two terms on the right-hand side are the familiar Reynolds stress and viscous dissipation terms. The last term represents the energy transfer between the 'odd' (u', v') and the 'even' (u'', v'') part of the flow.

If we consider a perturbation of $O(\epsilon)$ upon a parallel flow, then u'' represents the first harmonic, and the last term can be considered of $O(\epsilon^4)$ for the purposes of discussion. When the basic flow is assumed to have a periodic component of wave-number 2α and amplitude δ, this term is of $O(\epsilon^2\delta)$ and so can contribute more effectively to the growth of the u', v' components if $\delta \gg \epsilon^2$. Because the energy integral is of $O(\epsilon^2)$, the transfer between the basic periodic flow and the disturbance has an $O(\delta)$ effect upon the rate of increase of energy of the disturbance.

The energy of the disturbance is not affected to such a degree when we look at a disturbance with wave-number, say α, equal to that of the periodic flow. Then, the last term in (7.1) is of $O(\epsilon\delta^3)$, because $u' \sim O(\delta)$ and $u'' \sim O(\epsilon\delta)$. After the energy of the basic flow is subtracted out of the equation, the predominant term in the energy integral is of $O(\epsilon\delta)$. Hence the transfer between the basic periodic flow and the disturbance has only an $O(\delta^2)$ effect upon the rate of increase of disturbance energy for this case. The subharmonic wave which has a wave-number exactly half that of the basic periodic flow is therefore quite unique in being able to receive energy from the periodic flow; its amplification due to this transfer is an order of magnitude greater than any other wave. Our model presupposes that any energy extracted from the periodic flow will be compensated by additional energy transferred from the mean flow; hence it is reasonable to say that the subharmonic receives energy via the basic periodic flow.

The interaction discussed in case C took place between the periodic flow of $O(\delta)$ and wave-number 2α and two disturbances, of $O(\epsilon)$ and with wave-numbers α and 3α respectively. The energy-transfer term will still be of $O(\epsilon^2\delta)$, with reference to (7.1), but will now be composed of terms representing energy exchange between the two disturbances as well as terms representing the transfer of energy from the periodic flow to either disturbance. The transfer of energy to (or from) the disturbance with wave-number α is still of $O(\delta)$, relative to the rate of increase of energy of that disturbance. A similar statement holds for the coupling terms which represent energy exchange between the disturbances. However, the transfer of energy directly to or from the periodic flow to the disturbance with wave-number 3α is at least no larger than $O(\delta^2)$; hence we expect that energy exchange between the two disturbances might have a stabilizing influence on the disturbance with wave-number α and a destabilizing influence on the disturbance with wave-number 3α. These statements are in accordance with the numerical results of §6.

The interaction discussed for case B cannot be discussed directly by reference to (7.1). However, it is interesting to examine the distribution of vorticity for the disturbances involved in the interaction (cf. Michalke 1964, figure 9). Disturbances with wave-numbers less than that for maximum growth have a smaller peak in vorticity, which is located further away from the inflexion point, than that for the disturbance undergoing maximum amplification. The opposite statement holds for disturbances with wave-numbers greater than that for maximum growth ($\alpha = 0{\cdot}4446$). The interaction can be viewed as a means of exchanging vorticity between the disturbances. Any vorticity fed into a disturbance with α less than $0{\cdot}4446$ would tend to make it resemble more the most unstable disturbance and so destabilize it. On the other hand, vorticity fed into a disturbance with α greater than $0{\cdot}4446$ would tend to make it more similar to the neutral disturbance and so stabilize it. Case B, which involves two waves whose wave-numbers are on either side of the most unstable wave-number, seems to consist of two such contradictory processes which therefore tend to cancel out.

The importance of the periodic nature of the basic flow cannot, however, be overemphasized. As clearly shown by (7.1), it is this feature which suggests that the growth rate of a subharmonic disturbance may be altered considerably from the value predicted by a parallel flow analysis. Domm (1956) has attempted to explain the frequency drop observed in experiments on free shear layers on the basis that the initial disturbances can be represented by potential vortices which then tend, as he shows, to slip around each other. As pointed out to the present author by Dr J. T. Stuart, the classical stability analysis of a single row of vortices, as described by Lamb (1959, Art. 156), permits the definition of a most unstable disturbance, whose wavelength is equal to twice the distance between pairs of vortices (this is obtained by maximizing the growth rate given in equation (12) of Lamb's presentation). The point of view of this paper is that the behaviour of the vortices can be explained on the basis of their periodic nature; the behaviour of disturbances in free shear layers can be explained on the same basis, which does not require us to identify such disturbances as potential vortices. One important advantage of the present analysis is that a critical amplitude for the primary disturbance can be defined rather naturally. A second is that a representation of the flow is constructed in which both fundamental and subharmonic waves are present, as well as other waves which result from their mutual and self-interaction. As figure 1 indicates, a considerable peak in energy can occur at the fundamental frequency even after the subharmonic has emerged. Furthermore, a recent experimental investigation by Browand (whose results were received by the present author after this analysis had been completed) has revealed that a significant energy peak can occur at $\frac{3}{2}$'s of the fundamental frequency. The present theory predicts the existence of such a response through the non-resonating terms of ψ_{11} (cf. 4.16) and also through the higher-order resonant interaction mentioned near the end of §4.

Browand's results also indicate that the amplitude of the fundamental is relatively constant (which is essential for the present model) until the subharmonic has grown to comparable size, at which point the fundamental decreases appreciably. The present linear analysis is, of course, inapplicable to this region of large-amplitude growth, whose characteristics probably account for the intermittent nature of the generation of the subharmonic oscillation which Browand also observed. He also found a variation across the shear layer of the wave speed of the subharmonic oscillation, which is inexplicable on the basis of the present model. However, it is encouraging that Browand concluded that three-dimensional effects are not important in the region of subharmonic growth and that the critical amplitude for the primary oscillation, at which the subharmonic emerges, is about 10 % of the freestream velocity. This figure is in quite reasonable agreement with the prediction of the present analysis.

It is obvious that the numerical calculations are of great importance to the conclusions of this paper; for these, the author is greatly indebted to Prof. W. C. Reynolds, who was at the NPL during 1964–65 while on leave from Stanford University. The author would also like to thank Prof. Reynolds, Dr S. Rosenblat, Mr J. Watson and, especially, Dr J. T. Stuart for their comments concerning the paper. He would also like to acknowledge the many benefits which he has derived from his association with Dr Stuart. This work was completed while the author held a Senior Research Fellowship at the NPL and was carried out as part of the research programme of the Laboratory. The paper is published by permission of the Director of the Laboratory.

Appendix

The original purpose of this appendix was to relate the eigenvalues corresponding to spatially growing disturbances to those corresponding to temporally growing disturbances for the case of the inviscid, parallel flow given by

$$U_0 = 0{\cdot}5(1+\tanh y), \tag{2.14}$$

through application of the analysis of Gaster (1962).

Since the time of the original writing, however, Michalke (1965*b*) has published his results concerning the spatial instabilities, which results were obtained by direct numerical calculation. It was decided, however, to retain this section, first, to show that excellent results are obtained by means of Gaster's analysis for the flow (2.14) and, secondly, to discuss an anomalous case arising for the flow (2.13).

Gaster assumed that the perturbed flow can be described by a stream function of the form

$$\psi(x,y,t) = \psi_0(y)+\phi(y)\exp\{i(\alpha x-\beta t)\}, \tag{A 1}$$

where α and β are, in general, complex. Our problem is to relate the known results for $\alpha_i = 0$ to the desired values for complex α but $\beta_i = 0$. Gaster assumed that α and β are analytic functions of each other and therefore fulfil the Cauchy–Riemann equations

$$\frac{\partial\beta_r}{\partial\alpha_r} = \frac{\partial\beta_i}{\partial\alpha_i}, \quad \frac{\partial\beta_r}{\partial\alpha_i} = -\frac{\partial\beta_i}{\partial\alpha_r}. \tag{A 2}$$

He then integrated these equations from a state (T) where $\alpha_i = 0$ to a state (S) where $\beta_i = 0$ while maintaining α_r as constant and so obtained

$$\beta_i(T) = -\int_0^{\alpha_i(S)} \frac{\partial\beta_r}{\partial\alpha_r}\,d\alpha_i, \tag{A 3}$$

$$\beta_r(S)-\beta_r(T) = -\int_0^{\alpha_i(S)} \frac{\partial\beta_i}{\partial\alpha_r}\,d\alpha_i. \tag{A 4}$$

For the flow (2.14), we know that $\beta_i(T)_{\max}$ is 0·09485 (cf. Michalke (1964), table 1). We anticipate that α_i will be of the same order of magnitude and expand the integrands of (A 3) and (A 5) in a Taylor's series about $\alpha_i = 0$:

$$\frac{\partial\beta_r}{\partial\alpha_r} = \left(\frac{\partial\beta_r}{\partial\alpha_r}\right)_{\alpha_i=0} + \alpha_i\left(\frac{\partial^2\beta_r}{\partial\alpha_r\partial\alpha_i}\right)_{\alpha_i=0} + O(\alpha_i^2), \tag{A 5}$$

$$\frac{\partial\beta_i}{\partial\alpha_r} = \left(\frac{\partial\beta_i}{\partial\alpha_r}\right)_{\alpha_i=0} + \alpha_i\left(\frac{\partial^2\beta_i}{\partial\alpha_r\partial\alpha_i}\right)_{\alpha_i=0} + O(\alpha_i^2). \tag{A 6}$$

After using the relations (A 2) and integrating, we obtain

$$\beta_i(T) = -\alpha_i\left(\frac{\partial\beta_r}{\partial\alpha_r}\right)_{\alpha_i=0} + \frac{\alpha_i^2}{2}\left(\frac{\partial^2\beta_i}{\partial\alpha_r^2}\right)_{\alpha_i=0} + O(\alpha_i^3), \tag{A 7}$$

$$\beta_r(S) - \beta_r(T) = -\alpha_i\left(\frac{\partial\beta_i}{\partial\alpha_r}\right)_{\alpha_i=0} - \frac{\alpha_i^2}{2}\left(\frac{\partial^2\beta_r}{\partial\alpha_r^2}\right)_{\alpha_i=0} + O(\alpha_i^3). \tag{A 8}$$

For the special case of the flow (2.14), the group velocity is equal to the wave velocity, and we have

$$\left(\frac{\partial\beta_r}{\partial\alpha_r}\right)_{\alpha_i=0} = c_r = 0{\cdot}5, \quad \left(\frac{\partial^2\beta_r}{\partial\alpha_r^2}\right)_{\alpha_i=0} = 0, \tag{A 9}$$

where c_r is the wave velocity. The first term in the expansion (A 7) therefore gives the often quoted result

$$\alpha_i = -\beta_i(T)/c_r = -\alpha_r c_i/c_r. \tag{A 10}$$

We shall find the error involved in this result by calculating the effect of the $O(\alpha_i^2)$ term. We shall also calculate $\beta_r(S)$, which is given by

$$\beta_r(S) = \beta_r(T) - \alpha_i(\partial\beta_i/\partial\alpha_r)_{\alpha_i=0}, \tag{A 11}$$

up to $O(\alpha_i^3)$.

First, however, consider (A 7) when the parallel flow is given by

$$U_{00} = \tanh y. \tag{2.13}$$

With reference to figure 2, we then have

$$(\partial\beta_r/\partial\alpha_r) = 0, \quad (\partial^2\beta_i/\partial\alpha_r^2)_{\alpha_i=0} < 0. \tag{A 12}$$

Because $\beta_i(T)$ is greater than zero for α_r less than unity and greater than zero, the result is invalid when only terms of the second order are considered. Either α_i is of exceptionally large magnitude for this case or β and α are not analytic functions of each other.

Some indication that this latter conclusion is correct results from consideration of the stability of the discontinuous flow

$$y > 0, \quad U_0 = U_1; \quad y < 0, \quad U_0 = U_2, \tag{A 13}$$

where U_1 and U_2 are constants. This flow should serve as a model for (2.13) and (2.14) in the limit $\alpha_r \to 0$. If we assume the disturbed flow to be described by

$$\psi_j(x,y,t) = U_j + \phi_j(y)\exp\{i\alpha(x-ct)\} \quad (j = 1,2), \tag{A 14}$$

where α and c may be complex, the equation for $\phi_j(y)$ is given by

$$\phi_j'' - \alpha^2\phi_j = 0. \tag{A 15}$$

In order to have a solution which decays as $y \to \pm\infty$, we must have $|\alpha_r| > 0$ and

$$\phi_1 = A\,e^{-\alpha y}, \quad \phi_2 = B\,e^{\alpha y}, \tag{A 16}$$

where we take $\alpha_r > 0$ in order to be definite. By matching the pressure across the interface and ensuring that the expression for the deflexion of the interface is unique, the eigenvalue relation is

$$c = \tfrac{1}{2}(U_1 + U_2) \pm \tfrac{1}{2}i(U_1 - U_2), \tag{A 17}$$

which is, of course, the classical Helmholtz result for $\alpha_i = 0$.

If we now stipulate that the solution can grow or decay only with x, we impose the condition

$$\alpha_i c_r + \alpha_r c_i = 0. \tag{A 18}$$

Our solution will then be of the form

$$\psi_j(x,y,t) = U_j + \phi_j(y)\exp\left\{i\alpha_r\left[x - \left(c_r + \frac{c_i^2}{c_r}\right)t\right] + \frac{\alpha_r c_i}{c_r}x\right\}. \tag{A 19}$$

This is exactly the result that would be obtained from (A 10) and (A 11) for $c_r \neq 0$. In the case of the flow (2.13), there are no solutions of this type because (A 18) would require $\alpha_r = 0$ for $c_r = 0$. The boundary conditions could then not be met. Hence, it seems improbable that there are solutions which can grow with distance in the streamwise direction for the flow (2.13), at least in the limit $\alpha_r \rightarrow 0$.

Further insight may be gained by application of Lin's perturbation analysis (4.8) about the neutral solution, because that analysis does not seem limited only to the case of real α. For the flow

$$U_0(y) = U_s + \tanh y, \tag{A 20}$$

where U_s is a constant, the relation is, with reference to (4.10),

$$c = U_s - i(\alpha^2 - 1)/\pi. \tag{A 21}$$

We therefore have

$$\beta = U_s\alpha - i\alpha(\alpha^2 - 1)/\pi. \tag{A 22}$$

If we now take $\alpha = 1 + \alpha_1$, where α_1 may be complex but with $|\alpha_1| \ll 1$, we have

$$\beta - U_s = \alpha_1[U_s - (2i/\pi)] - (3i\alpha_1^2/\pi) - (i\alpha_1^3/\pi). \tag{A 23}$$

For the case $U_s = 0$ and $\beta_i = 0$, we have to first order

$$\alpha_1 = \tfrac{1}{2}i\pi\beta_r. \tag{A 24}$$

Thus, waves which travel in the positive x-direction ($\beta_r > 0$) will be damped in that direction, whereas waves which travel in the negative x-direction ($\beta_r < 0$) will also be damped in that direction. It therefore only seems reasonable to conclude that such waves are stable.

The first-order result involves no change in wave-number. By including higher-order terms, the dependence on wave-number can be established. We let

$$\alpha_1 = \tfrac{1}{2}i\pi\beta_r + a_2\beta_r^2 + a_3\beta_r^3 \tag{A 25}$$

and substitute into (A 23) for $U_s = 0$. We find that to third order

$$\alpha_1 = \tfrac{1}{2}i\pi\beta_r + \tfrac{3}{8}\pi^2\beta_r^2 - \tfrac{1}{2}i\pi^3\beta_r^3. \tag{A 26}$$

Thus, the real part of α_1 increases with β_r, regardless of the sign of β_r; i.e. the above solutions exist only for $\alpha_r > 1$. Solutions which grow with time, of course, exist only for $\alpha_r < 1$, while solutions which decay with time cannot be investigated on a purely inviscid basis. Whether or not the above solutions are valid limits to any viscous solutions would seem to be a worthy topic for further analysis. We note in passing that (A 26) seems valid only for very small values of β_r.

Now consider the case when $U_s > 0$. The first-order result is, for $\beta_i = 0$,

$$\alpha_{1,r} = \frac{(\beta_r - U_s)\,U_s}{U_s^2 + (4/\pi^2)}, \quad \alpha_{1,i} = \frac{(\beta_r - U_s)(2/\pi)}{U_s^2 + (4/\pi^2)}. \tag{A 27}$$

We then have $\alpha_{1,r}$ and $\alpha_{1,i}$ less than zero if β_r is less than U_s. Thus a small decrease in wave-number below the neutral value corresponds to a spatial growth rate such that the wave grows in the positive x-direction and to a frequency which is below the corresponding frequency of the temporally amplified solution. These results agree with the results gained from Gaster's analysis and which will now be discussed.

We shall calculate α_i for the flow given by (2.14) by means of (A 7) when terms of $O(\alpha_i^2)$ are included. If we define

$$\gamma = (\partial^2 \beta_i / \partial \alpha_r^2)_{\alpha_i = 0} \tag{A 28}$$

the relation for α_i is then

$$\alpha_i = c_r \pm (c_r^2 + 2\gamma\beta_i)^{\frac{1}{2}}/\gamma. \tag{A 29}$$

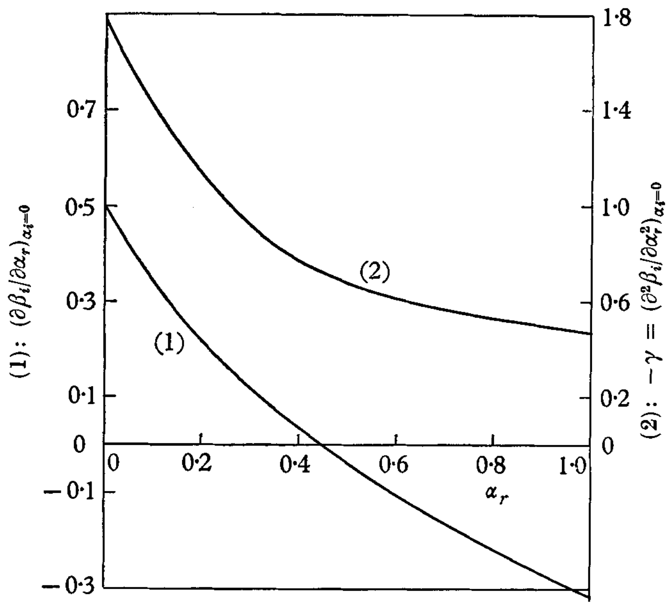

FIGURE 4. The derivatives of the temporal growth rate with respect to wave-number for the flow $U_0(y) = 0{\cdot}5(1 + \tanh y)$.

If we take the limit $\alpha_r \to 0$ ($\beta_i \to 0$), we obtain (A 19) by choosing the negative sign before the radical. The quantities $(\partial\beta_i/\partial\alpha_r)_{\alpha_i=0}$ and γ were obtained by linear interpolation from the results of Michalke (1964) and are shown in figure 4 for the flow (2.14). The asymptotic value of γ as $\alpha_r \to 0$ was obtained from the results of Drazin & Howard (1962), and the value of $(\partial\beta_i/\partial\alpha_r)_{\alpha_i=0}$ as $\alpha_r \to 1$ was obtained through application of the formula of Lin (4.8). The value of γ as predicted by this latter relation for $\alpha_r \to 1$ is about twice the value obtained by interpolation and was not used.

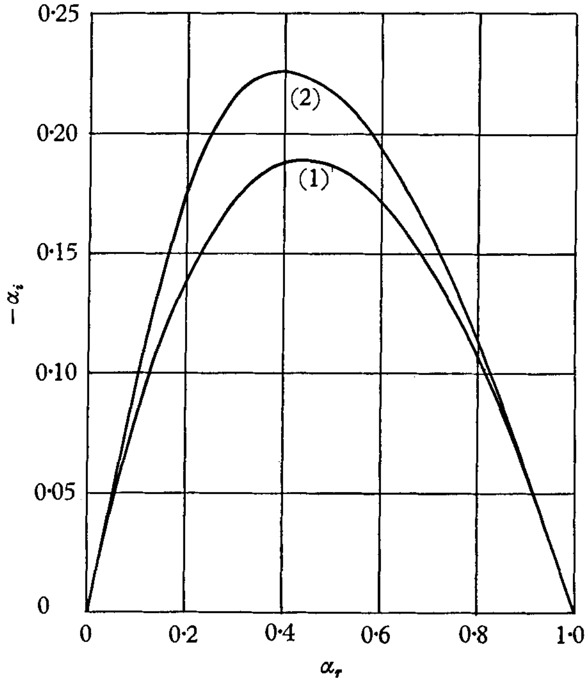

FIGURE 5. Spatial growth rate of disturbances to the flow $U_0(y) = 0{\cdot}5(1+\tanh y)$, according to (1) first-order theory (equation (A 10)) and (2) second-order theory (equation (A 21)).

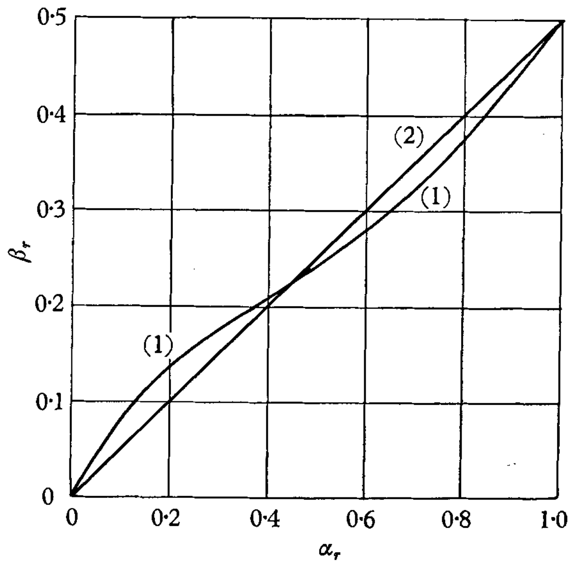

FIGURE 6. The frequency of disturbances to the flow $U_0(y) = 0{\cdot}5(1+\tanh y)$. (1), spatially growing disturbances. (2), temporally growing disturbances.

The values of α_i obtained from (A 21), with the negative sign chosen, for various α_r are shown in figure 5. The divergence from the exact results obtained by Michalke (1965*b*) is so small as to preclude comparison in this figure. The values predicted by (A 10) are also shown for purposes of comparison. It is clear that use of (A 10) involves an error of about 18 % for the maximum value of α_i as found by (A 21). This error presumably accounts for the fact that Sato (1959, figure 14) measured spatial amplification rates somewhat in excess of the theoretical value, as obtained from the temporal growth analysis of Lessen & Fox (1955) together with the transformation (A 10). The value of wave-number for maximum spatial amplification is also slightly less than that for maximum temporal amplification. The present results are correct up to $O(\alpha_i^4)$ because the next term in the expansion (A 7) is of the form $(\alpha_i^3/6)(\partial^3\beta_r/\partial\alpha_r^3)_{\alpha_i=0}$, which is zero for the flow (2.14).

The variation of $\beta_r(s)$ with α_r, obtained from (A 11) with use of the second-order estimate of α_i, is shown in figure 6 along with $\beta_r(T)$. It is clear that the frequencies of the spatially growing waves vary considerably from the linear dependence upon α_r which is characteristic of temporally growing waves. The calculations show that $\beta_r(s)$ is greater or less than $\beta_r(T)$ depending upon whether α_r is less or greater, respectively, than the wave-number for maximum temporal growth. Only for this special wave-number (along with the limiting cases $\alpha_r = 0$ and $\alpha_r = 1$) do the two frequencies coincide.

This dispersive feature of the spatially growing waves is the basis of the remarks made near the end of §4 concerning the inability of the spatially growing disturbances to satisfy the conditions (3.4), (3.5) for resonance.

For results concerning the relation between the two types of disturbance in the inviscid jet and wake, the reader is referred to the recent paper by Betchov & Criminale (1966).

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